Analyzing the Stochastic Stability of Neural Networks with Semi-Markov Jump Parameters

Lulu Zhang, Jiayue Sun, and Huaguang Zhang

***Abstract*—This paper addresses the issue of stochastic stability for continuous-time semi-Markovian jump neural networks. Initially, according to the characteristics of the time-delay interval, the Lyapunov-Krasovskii functional (LKF) with the semi-Markov process is constructed to ensure the stability of the switching networks. Next, the promoted double integral inequality lemma is utilized to estimate the weak infinitesimal operator of the designed LKF, and this paper establishes a time-delay correlation criterion. Moreover, the criterion is combined with the Lyapunov stability theory to make the system reach stability in the mean-square sense. Finally, the paper provides an example to demonstrate the effectiveness of the proposed approach.**

***Index Terms*—Neural networks (NNs), semi-Markov jump systems (sMJSs), sojourn-time (ST), stochastic stability analysis**

I. Introduction

The operational stability of neural networks (NNs) is often compromised by various external factors such as component damage, subsystem connection failure, and external interference, leading to abrupt changes in their structural and system parameters. Such sudden changes often exhibit random characteristics and follow certain statistical properties, including the Markov and semi-Markov properties. Therefore, these properties are essential for investigating the stability of stochastic neural networks [1, 2].

The Markov jump system (MJS) [3] is a specific kind of stochastic switching system that has been extensively studied in Refs. [4–8]. The sojourn-time (ST) of the MJS is a random variable that follows an exponential distribution function (EDF) [9, 10]. In this line, Wang et al. [11] analyzed the H_∞ control problem of continuous-time hidden Markov jump systems. In Ref. [12], the stability of continuous-time Markov jump piecewise-affine systems is analyzed. However, the EDF has certain limitations, such as the system being connected only with the last mode and the transition rate matrix of the MJS being time-invariant, which restrict its practical applications.

In contrast, the semi-Markov jump system (sMJS) follows a non-EDF for the ST, and the transition probability between different modes depends on not only the last mode but also the previous historical modes [13]. The sMJS relaxes the restrictions of the EDF and is useful for modeling more complex systems. Several studies have explored the use of sMJS in complex system modeling [14–16]. Recently, researchers have proposed ST-based stability criteria for the sliding-mode control (SMC) problem of nonlinear sMJS via the weak infinitesimal operator theory [17]. The adaptive SMC problem for sMJS has also been investigated in Ref. [18]. Therefore, sMJS has garnered widespread attention among scientists and researchers (see Refs. [19–21]). In terms of application, sMJS can be used to model complex and various stochastic systems, and its properties can be employed to analyze fault-tolerant control systems [22]. A specific practical application of sMJS can be seen in the cognitive radio network in Ref. [23], where it is precisely designed to model the stochastic behavior of each channel.

In view of the above consideration, this paper addressed the stability problem of semi-Markov jump system NNs that possesses time-delay. Specifically, utilizing the Lyapunov stability theory, an appropriate Lyapunov-Krasovskii functional (LKF) is designed. Then, we utilized inverted convex combination technology and generalized double integral inequality to derive the stability condition with time-delay and semi-Markov jump process by applying the weak infinitesimal operator approach. In the end, the obtained results are illustrated through the application of the numerical examples. Furthermore, Table 1 describes the symbols and meanings employed in this paper. If not declared beforehand, the matrices in this paper have appropriate dimensions.

II. Problem Description

In this paper, the considered sMJS model is designed as follows

$$\begin{cases} \dot{x}(t) = -D(m(t))x(t) + A(m(t))\tilde{k}(x(t)) + \\ \quad B(m(t))\tilde{k}(x(t-\omega(t))) + u(t), \\ x(t) = \tilde{\delta}(t),\ t \in [-\omega_2,\ 0],\ m(0) = m_0 \end{cases} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$ denotes the n-dimensional neuron state vector of Eq. (1). $m(t)$, $t \geqslant 0$, means the semi-Markov process (sMP). $\tilde{k}(x(t)) = \left[\tilde{k}_1(x_1(t)),\ \tilde{k}_2(x_2(t)), \ldots,\ \tilde{k}_n(x_n(t))\right]^{\mathrm{T}}$ signifies the activation function of the neuron. $\omega(t)$ represents the bounded delay function. $u(t)$ denotes the control input. $A(m(t))$, $B(m(t))$, and $D(m(t))$ denote neuron

Manuscript received: 24 March 2023; revised: 6 July 2023; accepted: 10 October 2023. *(Corresponding author: Huaguang Zhang.)*

Citation: L. Zhang, J. Sun, and H. Zhang, Analyzing the stochastic stability of neural networks with semi-Markov jump parameters, *IJICS*, 2024, 29(1), 12–20.

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Digital Object Identifier 10.62678/IJICS202403.10106

Table 1 Symbol description.

Symbol	Meaning
Q^{-1}	Matrix inverse
Q^{T}	Transpose of a matrix
$Q>0\ (<0)$	Positive (negative) definite symmetric matrix
0_n, $0_{m\times n}$, and I_n	n-dimensional zero matrix, $m\times n$-dimensional zero matrix, and n-dimensional identity matrix
$\alpha^{\mathrm{T}}Q(*)$ and $\alpha Q(*)^{\mathrm{T}}$	$\alpha^{\mathrm{T}}Q\alpha$ and $\alpha Q\alpha^{\mathrm{T}}$
$\mathrm{sym}\{Z\}$	$Z+Z^{\mathrm{T}}$
$\mathrm{diag}\{\cdot\}$	Diagonal matrix or block diagonal matrix
$\begin{bmatrix} A & B \\ * & C \end{bmatrix}$	$\begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix}$
$(\Omega,\mathcal{F},\mathcal{P})$	Given probability space, Ω denotes the sample space, $\mathcal{F}$ means a subset of the sample space, and $\mathcal{P}$ is the probability
$\varepsilon\{\cdot\}$	Mathematical expectation for probability
$\lVert\cdot\rVert$	Euclidean norm of a vector
$\dot{x}$	Derivative of x

weight matrix, and they are all matrix functions about sMP. $\tilde{\delta}(t)$ denotes the initial continuous function, where $t\in[-\omega_2,\ 0]$ and ω_2 is a positive constant. m_0 denotes model initial value.

The conclusions of this paper rely on Assumptions 1 and 2.

Assumption 1 The bounded delay function $\omega(t)$ satisfies the following conditions

$$\begin{aligned} &0\leqslant\omega_1\leqslant\omega(t)\leqslant\omega_2,\\ &\partial_1\leqslant\dot{\omega}(t)\leqslant\partial_2 \end{aligned}\tag{2}$$

where ω_1, ω_2, ∂_1, and $\partial_2>0$ are known constants.

Assumption 2 It is assumed that $\tilde{k}(x(t))$ is bounded and satisfies

$$0\leqslant\phi_i^-\leqslant\frac{\tilde{k}_i(s_1)-\tilde{k}_i(s_2)}{s_1-s_2}\leqslant\phi_i^+\tag{3}$$

where $s_1\neq s_2$, $i=1,2,\ldots,n$, ϕ_i^-, and ϕ_i^+ mean known constants.

Definitions 1 and 2, and Lemmas 1–3 are required.

Definition 1 Random processes $\{m_k\}$ take values on $\mathcal{S}=\{1,2,\ldots,\mathcal{N}\}$, m_k denotes state that when the mode changes at the k-th transition. And both random processes $\{t_k\}$ and $\{h_k\}$ have positive integer values, where t_k denotes time that when the mode changes at the k-th transition, $t_0=0$. h_k denotes ST that when the mode changes from the $(k-1)$-th transition to the k-th transition, $h_0=0$ and $h_k=t_k-t_{k-1}$. k is a positive integer [24, 25].

In $(\Omega,\mathcal{F},\mathcal{P})$, the sMP $\{m(t),h\}_{t>0}:=\{m_k,h_k\}$ with continuous-time on finite state spaces $\mathcal{S}=\{1,2,\ldots,\mathcal{N}\}$, in which the state transition rate (TR) matrix is as follows

$$\begin{aligned} &\mathrm{P_r}\{m_{k+1}=\eta,h_{k+1}\leqslant h+\sigma\mid m_k=\iota,h_{k+1}>h\}=\\ &\qquad\begin{cases}\pi_{\iota\eta}(h)\sigma+o(\sigma),\ \eta\neq\iota;\\ 1+\pi_{\iota\iota}(h)\sigma+o(\sigma),\ \eta=\iota\end{cases} \end{aligned}\tag{4}$$

where Pr represents the probability, and $\pi_{\iota\eta}(h)$ means the transition rate at time t jumping from mode ι to mode η at time $t+\sigma$, $\iota\neq\eta$, TR matrix $\tilde{\Pi}=[\pi_{\iota\eta}(h)]_{\mathcal{N}\times\mathcal{N}}$, and $\pi_{\iota\iota}(h)=-\sum_{\eta=1,\eta\neq\iota}^{\mathcal{N}}\pi_{\iota\eta}(h)$, where $\sigma>0$ is a constant and $o(\sigma)$ is defined by $\lim_{\sigma\to0}[o(\sigma)/\sigma]=0$.

This paper focuses on studying the stability of Eq. (1), and usually assuming that its equilibrium point x^* exists. Let $x^*=\left(x_1^*,x_2^*,\ldots,x_n^*\right)$ and $y(t)=x(t)-x^*$, so Eq. (1) is transformed into

$$\begin{cases}\dot{y}(t)=-D(m(t))y(t)+A(m(t))k(y(t))+B(m(t))\times\\ \quad k(y(t-\omega(t)))+u(t),\\ y(t)=\delta(t),\ t\in[-\omega_2,\ 0]\end{cases}\tag{5}$$

where $k(y(t))=\tilde{k}(y(t)+x^*)-\tilde{k}(x^*)$ and $\delta(t)$ is the initial function, therefore, the main purpose of this paper is changed from studying the stability of Eq. (1) to studying the stability of Eq. (5). According to Formula (3) and $k_i(0)=0$, one has

$$0\leqslant\phi_i^-\leqslant\frac{k_i(s_1)-k_i(s_2)}{s_1-s_2}\leqslant\phi_i^+,\ s_1\neq s_2\tag{6}$$

Definition 2 For any initial condition $\delta(t)$, $t\in[-\omega_2,\ 0]$ and $m_0\in\mathcal{S}$, which satisfy

$$\lim_{t\to\infty}\varepsilon\left\{\int_0^t\lVert x(s)\rVert^2\mathrm{d}s\mid(\delta,m_0)\right\}\leqslant\infty\tag{7}$$

Equation (1) is stochastic stable at the equilibrium point.

Lemma 1 Inverted convex combination technology For any vectors β_1 and β_2, symmetric matrix T, arbitrary matrix S, and constant α, $0\leqslant\alpha\leqslant1$, if satisfying $\begin{bmatrix}T & S\\ * & T\end{bmatrix}\geqslant0$, Formula (8) is correct [21]

$$\frac{1}{\alpha}\beta_1^{\mathrm{T}}T\beta_1+\frac{1}{1-\alpha}\beta_2^{\mathrm{T}}T\beta_2\geqslant\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}T & S\\ * & T\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}\tag{8}$$

Lemma 2 Generalized double integral inequality For any matrices X_1 and X_2, symmetric positive definite matrix Y, and two functions $\varsigma_1(t)$ and $\varsigma_2(t)$, which satisfy $0\leqslant\varsigma_1\leqslant\varsigma_1(t)\leqslant\varsigma_2(t)\leqslant\varsigma_2$, the vector function $z(s):[\varsigma_1,\varsigma_2]\to\mathbb{R}^n$, $\int_{\varsigma_1(t)}^{\varsigma_2(t)}\int_{\theta}^{\varsigma_2(t)}z(s)\mathrm{d}s\mathrm{d}\theta=\ell_1^{\mathrm{T}}\zeta_1(t)$, $\int_{\varsigma_1(t)}^{\varsigma_2(t)}\int_{\varsigma_1(t)}^{\theta}z(s)\mathrm{d}s\mathrm{d}\theta=\ell_2^{\mathrm{T}}\zeta_2(t)$, matrices ℓ_1 and $\ell_2\in\mathbb{R}^{k\times n}$, and $\zeta_1(t)$ and $\zeta_2(t)\in\mathbb{R}^k$, Formulas (9) and (10) are correct [26]

$$\begin{aligned}&\zeta_1^{\mathrm{T}}(t)\left[X_1\ell_1^{\mathrm{T}}+\ell_1X_1^{\mathrm{T}}-\frac{\varsigma_{12}^2(t)}{2}X_1Y^{-1}X_1^{\mathrm{T}}\right]\zeta_1(t)\leqslant\\ &\int_{\varsigma_1(t)}^{\varsigma_2(t)}\int_{\theta}^{\varsigma_2(t)}z^{\mathrm{T}}(s)Yz(s)\mathrm{d}s\mathrm{d}\theta\end{aligned}\tag{9}$$

$$\begin{aligned}&\zeta_2^{\mathrm{T}}(t)\left[X_2\ell_2^{\mathrm{T}}+\ell_2X_2^{\mathrm{T}}-\frac{\varsigma_{12}^2(t)}{2}X_2Y^{-1}X_2^{\mathrm{T}}\right]\zeta_2(t)\leqslant\\ &\int_{\varsigma_1(t)}^{\varsigma_2(t)}\int_{\varsigma_1(t)}^{\theta}z^{\mathrm{T}}(s)Yz(s)\mathrm{d}s\mathrm{d}\theta\end{aligned}\tag{10}$$

where $\varsigma_{12}(t)=\varsigma_2(t)-\varsigma_1(t)$.

Lemma 3 Assume that α_1 and β are real vectors with dimensions n_1 and n_2, given real symmetric positive definite matrices $\mathcal{A}_1\in\mathbb{R}^{n_1\times n_1}$ and $\mathcal{A}_2\in\mathbb{R}^{n_2\times n_2}$, if $\begin{bmatrix}\mathcal{A}_1 & S\\ * & \mathcal{A}_2\end{bmatrix}\geqslant0$, for any scalar $\kappa>0$, $S\in\mathbb{R}^{n_1\times n_2}$, Formula (11) is correct [27]

$$-2\alpha_1^{\mathrm{T}}S\beta\leqslant\kappa\alpha_1^{\mathrm{T}}\mathcal{A}_1\alpha_1+\kappa^{-1}\beta^{\mathrm{T}}\mathcal{A}_2\beta\tag{11}$$

III. MAIN RESULT

To obtain explicit expressions, Eq. (12) is given

$$\zeta(t)=\begin{bmatrix} y^{\mathrm{T}}(t),y^{\mathrm{T}}(t-\omega_1),y^{\mathrm{T}}(t-\omega(t)),y^{\mathrm{T}}(t-\omega_2), \\ \frac{1}{\omega(t)}\int_{t-\omega(t)}^{\mathrm{T}}y^{\mathrm{T}}(s)\mathrm{d}s,\frac{1}{\omega_2-\omega(t)}\int_{t-\omega_2}^{t-\omega(t)}y^{\mathrm{T}}(s)\mathrm{d}s, \\ \frac{1}{\omega(t)-\omega_1}\int_{t-\omega(t)}^{t-\omega_1}y^{\mathrm{T}}(s)\mathrm{d}s,k^{\mathrm{T}}(y(t)), \\ k^{\mathrm{T}}(y(t-\omega_1)),k^{\mathrm{T}}(y(t-\omega(t))), \\ k^{\mathrm{T}}(y(t-\omega_2)) \end{bmatrix}^{\mathrm{T}} \tag{12}$$

Remark 1 When $\omega(t)=0$ and $\omega_2=\omega(t)=\omega_1$, based on the integral median theorem, it follows that

$$\lim_{\omega(t)\to 0^+}\frac{1}{\omega(t)}\int_{t-\omega(t)}^{t}y(s)\mathrm{d}s=y(t) \tag{13}$$

$$\lim_{\omega(t)\to\omega_2^-}\frac{1}{\omega_2-\omega(t)}\int_{t-\omega_2}^{t-\omega(t)}y(s)\mathrm{d}s=y(t-\omega_2) \tag{14}$$

$$\lim_{\omega(t)\to\omega_1^-}\frac{1}{\omega(t)-\omega_1}\int_{t-\omega(t)}^{t-\omega_1}y(s)\mathrm{d}s=y(t-\omega_1) \tag{15}$$

Hence define

$$\frac{1}{\omega(t)}\int_{t-\omega(t)}^{t}y(s)\mathrm{d}s=y(t) \tag{16}$$

$$\frac{1}{\omega_2-\omega(t)}\int_{t-\omega_2}^{t-\omega(t)}y(s)\mathrm{d}s=y(t-\omega_2) \tag{17}$$

$$\frac{1}{\omega(t)-\omega_1}\int_{t-\omega(t)}^{t-\omega_1}y(s)\mathrm{d}s=y(t-\omega_1) \tag{18}$$

$$\frac{1}{\omega_2-\omega(t)}\int_{t-\omega_2}^{t-\omega(t)}\dot{y}(s)\mathrm{d}s=\dot{y}(t-\omega_2) \tag{19}$$

For simplicity, we define the following representations

$$\varGamma_1=[-D(m(t)),0_{n\times 6n},A(m(t)),B(m(t)),0_{n\times 2n}] \tag{20}$$

$$e_l=[0_{n\times(l-1)n},I_n,\ 0_{n\times(11-l)n}],\ l=1,2,3,...,11 \tag{21}$$

$$\varpi_r=[e_r,e_{r+7}],\ r=1,2,3,4 \tag{22}$$

$$\mathcal{Y}_1=[\varPi_1,\varPi_2],\ \mathcal{Y}_2=[\varPi_2,\varPi_3] \tag{23}$$

$$\psi_5=e_2-e_3,\ \psi_6=e_3-e_4 \tag{24}$$

$$\varPi_1=[e_1-e_3,e_1+e_3-2e_5] \tag{25}$$

$$\varPi_2=[e_3-e_4,e_3+e_4-2e_6] \tag{26}$$

$$\varPi_3=[e_2-e_3,e_2+e_3-2e_7] \tag{27}$$

$$L_1=\mathrm{diag}\left\{\phi_1^-\phi_1^+,\phi_2^-\phi_2^+,\ldots,\phi_n^-\phi_n^+\right\} \tag{28}$$

$$L_2=\mathrm{diag}\left\{\phi_1^-+\phi_1^+,\phi_2^-+\phi_2^+,\ldots,\phi_n^-+\phi_n^+\right\} \tag{29}$$

The results of the stability analysis of Eq. (5) with $u(t)\equiv 0$ are given in the rest of this section.

Theorem 1 According to Assumption 1, for given scalars $\omega_1>0$, $\omega_2>0$, $\partial_1>0$, and $\partial_2>0$, if there exist matrices $P(m(t))>0$, $\mathcal{J}_i(m(t))>0$, $Q_i>0$, $i=1,\ 2,\ 3$, $O_j(m(t))>0$, $\varTheta_j>0$, $\Re_j>0$, $j=1,\ 2$, positive definite matrices G_1 and G_2, positive diagonal matrices L_1 and L_2, and real matrices of appropriate dimension U_1, U_2, F_1, F_2, X, Y, and S, Formulas (30)–(34) hold

$$\begin{bmatrix}\tilde{Z}_1 & X\\ * & \tilde{Z}_1\end{bmatrix}>0,\begin{bmatrix}\tilde{Z}_2 & Y\\ * & \tilde{Z}_2\end{bmatrix}>0,\begin{bmatrix}G_1 & S\\ * & G_2\end{bmatrix}\geqslant 0 \tag{30}$$

$$\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)\mathcal{J}_\mu(\eta)\leqslant O_\mu(\iota),\ \mu=1,2 \tag{31}$$

$$\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)\sum_{u=2}^{3}\mathcal{J}_u(\eta)\leqslant O_2(\iota) \tag{32}$$

$$\sum_{\eta=1}^{\mathcal{N}}\omega_\mu\pi_{\iota\eta}(h)O_\mu(\eta)\leqslant\Re_\mu,\ \mu=1,2 \tag{33}$$

$$\begin{cases}\Im(\omega(t),\dot{\omega}(t))\big|_{\omega(t)=\omega_1,\dot{\omega}(t)=\partial_1}<0,\\ \Im(\omega(t),\dot{\omega}(t))\big|_{\omega(t)=\omega_1,\dot{\omega}(t)=\partial_2}<0,\\ \Im(\omega(t),\dot{\omega}(t))\big|_{\omega(t)=\omega_2,\dot{\omega}(t)=\partial_1}<0,\\ \Im(\omega(t),\dot{\omega}(t))\big|_{\omega(t)=\omega_2,\dot{\omega}(t)=\partial_2}<0\end{cases} \tag{34}$$

where matrices $\tilde{Z}_1$ and $\tilde{Z}_2$ satisfy $\tilde{Z}_1=\mathrm{diag}\{\varTheta_1,\ 3\varTheta_1\}$ and $\tilde{Z}_2=\mathrm{diag}\{\varTheta_2,\ 3\varTheta_2\}$, and

$$\Im(\omega(t),\dot{\omega}(t))=\begin{bmatrix}\Im_1(\omega(t),\dot{\omega}(t)) & \Im_2(\omega(t),\dot{\omega}(t))\\ * & -\Im_3(\omega(t),\dot{\omega}(t))\end{bmatrix} \tag{35}$$

$$\Im_1(\omega(t),\dot{\omega}(t))=\tilde{\Im}_1(\omega(t),\dot{\omega}(t))+\tilde{\Im}_2(\omega(t),\dot{\omega}(t))+ \tilde{\Im}_3(\omega(t),\dot{\omega}(t))+\tilde{\Im}_4(\omega(t),\dot{\omega}(t)) \tag{36}$$

$$\begin{aligned}\tilde{\Im}_1(\omega(t),\dot{\omega}(t))=&\ \mathrm{sym}\left\{e_1^{\mathrm{T}}\varGamma_1\right\}+e_1^{\mathrm{T}}\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)P(\eta)e_1+\\ &e_1^{\mathrm{T}}(\mathcal{J}_1(\iota)+\mathcal{J}_2(\iota)+\mathcal{J}_3(\iota)+\omega_1O_1(\iota)+\\ &\omega_1\Re_1+\omega_2O_2(\iota)+\omega_2\Re_2)e_1-\\ &e_2^{\mathrm{T}}\mathcal{J}_1(\iota)e_2-e_4^{\mathrm{T}}\mathcal{J}_2(\iota)e_4-(1-\dot{\omega}(t))e_3^{\mathrm{T}}\times\\ &\mathcal{J}_3(\iota)e_3+\varpi_1^{\mathrm{T}}Q_1\varpi_1-\varpi_2^{\mathrm{T}}Q_1\varpi_2+\varpi_2^{\mathrm{T}}\times\\ &Q_2\varpi_2-(1-\dot{\omega}(t))\varpi_3^{\mathrm{T}}Q_2\varpi_3+(1-\dot{\omega}(t))\times\\ &\varpi_3^{\mathrm{T}}Q_3\varpi_3-\varpi_4^{\mathrm{T}}Q_3\varpi_4+\omega_2\varGamma_1^{\mathrm{T}}\varTheta_1\varGamma_1+\\ &(\omega_2-\omega_1)\varGamma_1^{\mathrm{T}}\varTheta_2\varGamma_1+\frac{1}{2}(\omega_2-\omega_1)^2\varGamma_1^{\mathrm{T}}\times\\ &(G_1+G_2)\varGamma_1+2\psi_5S\psi_6^{\mathrm{T}}\end{aligned} \tag{37}$$

$$\begin{aligned}\tilde{\Im}_2(\omega(t),\dot{\omega}(t))=&\ U_1\psi_1^{\mathrm{T}}(\omega(t))+\psi_1(\omega(t))U_1^{\mathrm{T}}+\\ &U_2\psi_2^{\mathrm{T}}(\omega(t))+\psi_2(\omega(t))U_2^{\mathrm{T}}+F_1\psi_3^{\mathrm{T}}(\omega(t))+\\ &\psi_3(\omega(t))F_1^{\mathrm{T}}+F_2\psi_4^{\mathrm{T}}(\omega(t))+\psi_4(\omega(t))F_2^{\mathrm{T}}\end{aligned} \tag{38}$$

$$\begin{aligned}\tilde{\Im}_3(\omega(t),\dot{\omega}(t))=&\ \mathrm{sym}\{-e_8^{\mathrm{T}}\aleph e_8+e_1^{\mathrm{T}}L_2\aleph e_8-e_1^{\mathrm{T}}L_1\aleph e_1-\\ &e_{10}^{\mathrm{T}}\tilde{\aleph}e_{10}+e_3^{\mathrm{T}}L_2\tilde{\aleph}e_{10}-e_3^{\mathrm{T}}L_1\tilde{\aleph}e_3\}\end{aligned} \tag{39}$$

$$\tilde{\Im}_4(\omega(t),\dot{\omega}(t))=-\frac{1}{\omega_2}\mathcal{Y}_1\varLambda_1\mathcal{Y}_1^{\mathrm{T}}-\frac{1}{\omega_2-\omega_1}\mathcal{Y}_2\varLambda_2\mathcal{Y}_2^{\mathrm{T}} \tag{40}$$

$$\begin{aligned}\Im_2(\omega(t),\dot{\omega}(t))=&\ [(\omega_2-\omega(t))U_1,(\omega(t)-\omega_1)U_2,\\ &(\omega_2-\omega(t))F_1,(\omega(t)-\omega_1)F_2]\end{aligned} \tag{41}$$

$$\Im_3(\omega(t),\dot{\omega}(t))=2\,\mathrm{diag}\{G_1,G_1,G_2,G_2\} \tag{42}$$

where $\psi_1^{\mathrm{T}}(\omega(t))=(\omega_2-\omega(t))(e_3-e_6)$, $\psi_2^{\mathrm{T}}(\omega(t))=(\omega(t)-\omega_1)(e_2-e_7)$, $\psi_3^{\mathrm{T}}(\omega(t))=(\omega_2-\omega(t))(e_6-e_4)$, $\psi_4^{\mathrm{T}}(\omega(t))=(\omega(t)-\omega_1)(e_7-e_3)$, $\varLambda_1=\begin{bmatrix}\tilde{Z}_1 & X\\ * & \tilde{Z}_1\end{bmatrix}$, and $\varLambda_2=\begin{bmatrix}\tilde{Z}_2 & Y\\ * & \tilde{Z}_2\end{bmatrix}$, respectively. $\aleph$ and $\tilde{\aleph}$ represent matrices.

Then, Eq. (5) with Formula (2) is stochastic stable in mean square at the equilibrium point.

Proof To prove the stability of Eq. (5) stimulated by Ref. [21], we design the following LKF $V(y(t), m(t))$

$$V(y(t),m(t)) = \sum_{g=1}^{2} V_g(y(t),m(t)) + \sum_{\bar{g}=3}^{5} V_{\bar{g}}(y(t),t) \tag{43}$$

where

$$V_1(y(t),m(t)) = y^{\mathrm{T}}(t)P(m(t))y(t) \tag{44}$$

$$\begin{aligned} V_2(y(t),m(t)) = & \int_{t-\omega_1}^{t} y^{\mathrm{T}}(s)\mathcal{J}_1(m(t))y(s)\mathrm{d}s + \\ & \int_{t-\omega_2}^{t} y^{\mathrm{T}}(s)\mathcal{J}_2(m(t))y(s)\mathrm{d}s + \\ & \int_{t-\omega(t)}^{t} y^{\mathrm{T}}(s)\mathcal{J}_3(m(t))y(s)\mathrm{d}s + \\ & \int_{-\omega_1}^{0}\int_{t+\theta}^{t} y^{\mathrm{T}}(s)O_1(m(t))y(s)\mathrm{d}s\mathrm{d}\theta + \\ & \int_{-\omega_2}^{0}\int_{t+\theta}^{t} y^{\mathrm{T}}(s)O_2(m(t))y(s)\mathrm{d}s\mathrm{d}\theta + \\ & \int_{-\omega_1}^{0}\int_{t+\theta}^{t} y^{\mathrm{T}}(s)\Re_1 y(s)\mathrm{d}s\mathrm{d}\theta + \\ & \int_{-\omega_2}^{0}\int_{t+\theta}^{t} y^{\mathrm{T}}(s)\Re_2 y(s)\mathrm{d}s\mathrm{d}\theta \end{aligned} \tag{45}$$

$$\begin{aligned} V_3(y(t),t) = & \int_{t-\omega_1}^{t} \eta^{\mathrm{T}}(s)Q_1\eta(s)\mathrm{d}s + \int_{t-\omega(t)}^{t-\omega_1} \eta^{\mathrm{T}}(s)Q_2\eta(s)\mathrm{d}s + \\ & \int_{t-\omega_2}^{t-\omega(t)} \eta^{\mathrm{T}}(s)Q_3\eta(s)\mathrm{d}s \end{aligned} \tag{46}$$

$$\begin{aligned} V_4(y(t),t) = & \int_{-\omega_2}^{0}\int_{t+\theta}^{t} \dot{y}^{\mathrm{T}}(s)\Theta_1\dot{y}(s)\mathrm{d}s\mathrm{d}\theta + \\ & \int_{-\omega_2}^{-\omega_1}\int_{t+\theta}^{t} \dot{y}^{\mathrm{T}}(s)\Theta_2\dot{y}(s)\mathrm{d}s\mathrm{d}\theta \end{aligned} \tag{47}$$

$$\begin{aligned} V_5(y(t),t) = & \int_{-\omega_2}^{-\omega_1}\int_{\theta}^{-\omega_1}\int_{t+s}^{t} \dot{y}^{\mathrm{T}}(u)G_1\dot{y}(u)\mathrm{d}u\mathrm{d}s\mathrm{d}\theta + \\ & \int_{-\omega_2}^{-\omega_1}\int_{-\omega_2}^{\theta}\int_{t+s}^{t} \dot{y}^{\mathrm{T}}(u)G_2\dot{y}(u)\mathrm{d}u\mathrm{d}s\mathrm{d}\theta \end{aligned} \tag{48}$$

Based on Eq. (5), the purpose is to calculate the weak infinitesimal operator of $V(y(t),m(t))$. The following will introduce the definition of the weak infinitesimal operator $\mathcal{L}$

$$\begin{aligned} & \mathcal{L}V(y(t),m(t)) = \\ & \lim_{\sigma\to 0^+} \frac{1}{\sigma}[\varepsilon\{V(y(t+\sigma),m(t+\sigma)) \mid y(t),m(t)\} - \\ & V(y(t),m(t))] \end{aligned} \tag{49}$$

where $\sigma > 0$ denotes a sufficiently small constant, $m(t) = \iota$, and $m(t+\sigma) = \eta$. In the light of Ref. [28] utilizing the full probability equation

$$\begin{aligned} & \mathcal{L}V_1(y(t),m(t)) = \\ & \lim_{\sigma\to 0^+} \frac{1}{\sigma}\left\{\sum_{\eta=1}^{\mathcal{N}} p_{\iota\eta}(t,\sigma)V_1(y(t+\sigma),m(t+\sigma)) - \right. \\ & \left. V_1(y(t),m(t))\right\},\ \eta \neq \iota = \\ & \lim_{\sigma\to 0^+} \frac{1}{\sigma}\left\{\sum_{\eta=1,\eta\neq\iota}^{\mathcal{N}} p_{\iota\eta}(t,\sigma)y^{\mathrm{T}}(t+\sigma)P(m(t+\sigma))y(t+\sigma) + \right. \\ & \left. p_{\iota\iota}(t,\sigma)y^{\mathrm{T}}(t+\sigma)P(m(t))y(t+\sigma) - y^{\mathrm{T}}(t)P(m(t))y(t)\right\} \end{aligned} \tag{50}$$

where ι, $\eta \in \mathcal{S}$, and

$$p_{\iota\eta}(t,\sigma) = q_{\iota\eta}\frac{\Delta_\iota(h+\sigma) - \Delta_\iota(h)}{1-\Delta_\iota(h)},$$

$$p_{\iota\iota}(t,\sigma) = \frac{1-\Delta_\iota(h+\sigma)}{1-\Delta_\iota(h)} \tag{51}$$

where $\Delta_\iota(h)$ is the cumulative distribution function (CDF) of h when system remains in the mode ι, and $q_{\iota\eta}$ is probability density function of system jumping from mode ι to mode η.

Remark 2 According to properties of CDF and Taylor's formula, via simple calculation, we get

$$\begin{aligned} & \sigma > 0,\ \lim_{\sigma\to 0}\frac{o(\sigma)}{\sigma} = 0, \\ & \lim_{\sigma\to 0^+}\frac{\Delta_\iota(h+\sigma) - \Delta_\iota(h)}{\sigma(1-\Delta_\iota(h))} = \pi_\iota(h), \\ & \lim_{\sigma\to 0^+}\frac{\Delta_\iota(h+\sigma) - \Delta_\iota(h)}{1-\Delta_\iota(h)} = 0, \\ & \lim_{\sigma\to 0^+}\frac{1-\Delta_\iota(h+\sigma)}{1-\Delta_\iota(h)} = 1 \end{aligned} \tag{52}$$

Similarly, due to the Taylor's formula, an approximation of $y(t+\sigma)$ can be expressed as $y(t+\sigma) = y(t) + \dot{y}(t)\sigma + o(\sigma)$, then, substituting an approximation of $y(t+\sigma)$ into Eq. (50), we have

$$\begin{aligned} & \mathcal{L}V_1(y(t),m(t)) = \\ & \lim_{\sigma\to 0^+}\frac{1}{\sigma}\sum_{\eta=1,\eta\neq\iota}^{\mathcal{N}} q_{\iota\eta}\frac{\Delta_\iota(h+\sigma)-\Delta_\iota(h)}{1-\Delta_\iota(h)}y^{\mathrm{T}}(t)P(\iota)y(t) - \\ & \lim_{\sigma\to 0^+}\frac{1}{\sigma}\frac{\Delta_\iota(h+\sigma)-\Delta_\iota(h)}{1-\Delta_\iota(h)}y^{\mathrm{T}}(t)P(\iota)y(t) + \\ & \lim_{\sigma\to 0^+}\frac{1}{\sigma}[\sigma\frac{1-\Delta_\iota(h+\sigma)}{1-\Delta_\iota(h)}y^{\mathrm{T}}(t)P(\iota)\dot{y}(t) + \\ & \sigma\frac{1-\Delta_\iota(h+\sigma)}{1-\Delta_\iota(h)}\dot{y}^{\mathrm{T}}(t)P(\iota)y(t)] \end{aligned} \tag{53}$$

along Eq. (52), we obtain

$$\begin{aligned} & \mathcal{L}V_1(y(t),m(t)) = \\ & \sum_{\eta=1,\eta\neq\iota}^{\mathcal{N}} q_{\iota\eta}\pi_\iota(h)y^{\mathrm{T}}(t)P(m(t))y(t) - \\ & \pi_\iota(h)y^{\mathrm{T}}(t)P(m(t))y(t) + \mathrm{sym}\left\{y^{\mathrm{T}}(t)P(m(t))\dot{y}(t)\right\} = \\ & \sum_{\eta=1}^{\mathcal{N}} \pi_{\iota\eta}(h)y^{\mathrm{T}}(t)P(m(t))y(t) + \mathrm{sym}\left\{y^{\mathrm{T}}(t)P(m(t))\dot{y}(t)\right\} = \\ & \zeta^{\mathrm{T}}(t)\left\{\mathrm{sym}\left\{e_1^{\mathrm{T}}P(\iota)\Gamma_1\right\} + e_1^{\mathrm{T}}\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)P(\eta)e_1\right\}\zeta(t) \end{aligned} \tag{54}$$

where $\pi_{\iota\eta}(h) = q_{\iota\eta}\pi_\iota(h)$ and $q_{\iota\iota} = -1$.

Inspired by Ref. [29], setting $\mathcal{G}(t,\mathcal{J}_1(m(t))) = \int_{t-\omega_1}^{t} y^{\mathrm{T}}(s)\times Q_1(m(t))y(s)\mathrm{d}s$, Eq. (55) is easily verified on the basis of Eq. (49)

$$\begin{aligned}&\mathcal{L}\mathcal{G}(t,\mathcal{J}_1(m(t)))=\\&\lim_{\sigma\to0^+}\frac{1}{\sigma}[\varepsilon\{\mathcal{G}(t+\sigma,\mathcal{J}_1(m(t+\sigma)))\mid m(t)=\iota\}-\\&\mathcal{G}(t,\mathcal{J}_1(m(t)))]=\\&\sum_{\eta=1,\eta\neq\iota}^{\mathcal{N}}\mathcal{G}(t,\mathcal{J}_1(m(t+\sigma)))+\dot{\mathcal{G}}(t,\mathcal{J}_1(m(t)))=\\&\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)\int_{t-\omega_1}^{t}y^{\mathrm{T}}(s)\mathcal{J}_1(\eta)y(s)\mathrm{d}s+\\&\left[y^{\mathrm{T}}(t)\mathcal{J}_1(\iota)y(t)-y^{\mathrm{T}}(t-\omega_1)\mathcal{J}_1(\iota)y(t-\omega_1)\right]\end{aligned}\tag{55}$$

Therefore, we attain weak infinitesimal operator of $V_2(y(t), m(t))$

$$\begin{aligned}&\mathcal{L}V_2(y(t),m(t))=\int_{t-\omega_1}^{t}y^{\mathrm{T}}(s)\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)\mathcal{J}_1(\eta)y(s)\mathrm{d}s+\\&\int_{t-\omega(t)}^{t}y^{\mathrm{T}}(s)\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)\sum_{u=2}^{3}\mathcal{J}_u(\eta)y(s)\mathrm{d}s+\\&\int_{t-\omega_2}^{t-\omega(t)}y^{\mathrm{T}}(s)\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)\mathcal{J}_2(\eta)y(s)\mathrm{d}s-\int_{t-\omega_1}^{t}y^{\mathrm{T}}(s)O_1(\iota)y(s)\mathrm{d}s-\\&\int_{t-\omega(t)}^{t}y^{\mathrm{T}}(s)O_2(\iota)y(s)\mathrm{d}s-\int_{t-\omega_2}^{t-\omega(t)}y^{\mathrm{T}}(s)O_2(\iota)y(s)\mathrm{d}s+\\&\int_{-\omega_1}^{0}\int_{t+\theta}^{t}y^{\mathrm{T}}(s)\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)O_1(\eta)y(s)\mathrm{d}s\mathrm{d}\theta-\\&\int_{t-\omega_1}^{t}y^{\mathrm{T}}(s)\Re_1y(s)\mathrm{d}s+\int_{-\omega_2}^{0}\int_{t+\theta}^{t}y^{\mathrm{T}}(s)\times\\&\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)O_2(\eta)y(s)\mathrm{d}s\mathrm{d}\theta-\int_{t-\omega_2}^{t}y^{\mathrm{T}}(s)\Re_2y(s)\mathrm{d}s+\\&\zeta^{\mathrm{T}}(t)\{e_1^{\mathrm{T}}(\mathcal{J}_1(\iota)+\mathcal{J}_2(\iota)+\mathcal{J}_3(\iota)+\omega_1O_1(\iota)+\\&\omega_1\Re_1+\omega_2O_2(\iota)+\omega_2\Re_2)e_1-e_2^{\mathrm{T}}\mathcal{J}_1(\iota)e_2-\\&e_4^{\mathrm{T}}\mathcal{J}_2(\iota)e_4(1-\dot{\omega}(t))e_3^{\mathrm{T}}\mathcal{J}_3(\iota)e_3\}\zeta(t)\end{aligned}\tag{56}$$

Altering the order of the integral terms derives

$$\begin{aligned}&\int_{-\omega_1}^{0}\int_{t+\theta}^{t}y^{\mathrm{T}}(s)\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)O_1(\eta)y(s)\mathrm{d}s\mathrm{d}\theta=\\&\int_{t-\omega_1}^{t}(s-t+\omega_1)y^{\mathrm{T}}(s)\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)O_1(\eta)y(s)\mathrm{d}s\leqslant\\&\int_{t-\omega_1}^{t}y^{\mathrm{T}}(s)\sum_{\eta=1}^{\mathcal{N}}\omega_1\pi_{\iota\eta}(h)O_1(\eta)y(s)\mathrm{d}s\end{aligned}\tag{57}$$

Similarly, Formula (58) is also true

$$\begin{aligned}&\int_{-\omega_2}^{0}\int_{t+\theta}^{t}y^{\mathrm{T}}(s)\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)O_2(\eta)y(s)\mathrm{d}s\mathrm{d}\theta\leqslant\\&\int_{t-\omega_2}^{t}y^{\mathrm{T}}(s)\sum_{\eta=1}^{\mathcal{N}}\omega_2\pi_{\iota\eta}(h)O_2(\eta)y(s)\mathrm{d}s\end{aligned}\tag{58}$$

The LKF weak infinitesimal operator without $m(t)$ can be calculated by

$$\begin{aligned}&\mathcal{L}V_3(y(t),t)=\\&\eta^{\mathrm{T}}(t)Q_1\eta(t)-\eta^{\mathrm{T}}(t-\omega_1)Q_1\eta(t-\omega_1)+\\&\eta^{\mathrm{T}}(t-\omega_1)Q_2\eta(t-\omega_1)-(1-\dot{\omega}(t))\eta^{\mathrm{T}}(t-\omega(t))Q_2\times\\&\eta(t-\omega(t))+(1-\dot{\omega}(t))\eta^{\mathrm{T}}(t-\omega(t))Q_3\eta(t-\omega(t))-\\&\eta^{\mathrm{T}}(t-\omega_2)Q_3\eta(t-\omega_2)=\\&\zeta^{\mathrm{T}}(t)[[e_1,e_8]Q_1[*]^{\mathrm{T}}-[e_2,e_9]Q_1[*]^{\mathrm{T}}+\\&[e_2,e_9]Q_2[*]^{\mathrm{T}}-(1-\dot{\omega}(t))[e_3,e_{10}]Q_2[*]^{\mathrm{T}}+\\&(1-\dot{\omega}(t))[e_3,e_{10}]Q_3[*]^{\mathrm{T}}-[e_4,e_{11}]Q_3[*]^{\mathrm{T}}]\zeta(t)=\\&\zeta^{\mathrm{T}}(t)[\varpi_1Q_1\varpi_1^{\mathrm{T}}-\varpi_2Q_1\varpi_2^{\mathrm{T}}+\varpi_2Q_2\varpi_2^{\mathrm{T}}-\\&(1-\dot{\omega}(t))\varpi_3Q_2\varpi_3^{\mathrm{T}}+(1-\dot{\omega}(t))\varpi_3Q_3\varpi_3^{\mathrm{T}}-\varpi_4Q_3\varpi_4^{\mathrm{T}}]\zeta(t)\end{aligned}\tag{59}$$

$$\begin{aligned}&\mathcal{L}V_4(y(t),t)=\\&\omega_2\dot{y}^{\mathrm{T}}(t)\Theta_1\dot{y}(t)-\int_{t-\omega_2}^{t}\dot{y}^{\mathrm{T}}(s)\Theta_1\dot{y}(s)\mathrm{d}s+\\&(\omega_2-\omega_1)\dot{y}^{\mathrm{T}}(t)\Theta_2\dot{y}(t)-\int_{t-\omega_2}^{t-\omega_1}\dot{y}^{\mathrm{T}}(s)\Theta_2\dot{y}(s)\mathrm{d}s=\\&\zeta^{\mathrm{T}}(t)\left[\omega_2\Gamma_1^{\mathrm{T}}\Theta_1\Gamma_1+(\omega_2-\omega_1)\Gamma_1^{\mathrm{T}}\Theta_2\Gamma_1\right]\zeta(t)-\\&\int_{t-\omega_2}^{t}\dot{y}^{\mathrm{T}}(s)\Theta_1\dot{y}(s)\mathrm{d}s-\int_{t-\omega_2}^{t-\omega_1}\dot{y}^{\mathrm{T}}(s)\Theta_2\dot{y}(s)\mathrm{d}s\end{aligned}\tag{60}$$

For $\omega_1 < \omega(t) < \omega_2$, utilizing the Wirtinger integral inequality (see Ref. [30]) and Lemma 1 to Θ_1 and Θ_2 dependent integral terms, we get

$$\begin{aligned}&-\int_{t-\omega_2}^{t}\dot{y}^{\mathrm{T}}(s)\Theta_1\dot{y}(s)\mathrm{d}s=-\int_{t-\omega(t)}^{t}\dot{y}^{\mathrm{T}}(s)\Theta_1\dot{y}(s)\mathrm{d}s-\\&\int_{t-\omega_2}^{t-\omega(t)}\dot{y}^{\mathrm{T}}(s)\Theta_1\dot{y}(s)\mathrm{d}s\leqslant\zeta^{\mathrm{T}}(t)\Bigg\{\frac{1}{\omega(t)}[e_1-e_3,e_1+\\&e_3-2e_5]\mathrm{diag}\{\Theta_1,3\Theta_1\}[*]^{\mathrm{T}}+\frac{1}{\omega_2-\omega(t)}[e_3-\\&e_4,e_3+e_4-2e_6]\mathrm{diag}\{\Theta_1,3\Theta_1\}[*]^{\mathrm{T}}\Bigg\}\zeta(t)\leqslant\\&-\zeta^{\mathrm{T}}(t)\left[\frac{1}{\omega(t)}\Pi_1\tilde{Z}_1\Pi_1^{\mathrm{T}}+\frac{1}{\omega_2-\omega(t)}\Pi_2\tilde{Z}_1\Pi_2^{\mathrm{T}}\right]\zeta(t)\leqslant\\&\frac{1}{\omega_2}\zeta^{\mathrm{T}}(t)\mathcal{Y}_1\Lambda_1\mathcal{Y}_1^{\mathrm{T}}\zeta(t)\end{aligned}\tag{61}$$

$$\begin{aligned}&\int_{t-\omega_2}^{t-\omega_1}\dot{y}^{\mathrm{T}}(s)\Theta_2\dot{y}(s)\mathrm{d}s=\\&\int_{t-\omega(t)}^{t-\omega_1}\dot{y}^{\mathrm{T}}(s)\Theta_2\dot{y}(s)\mathrm{d}s-\int_{t-\omega_2}^{t-\omega(t)}\dot{y}^{\mathrm{T}}(s)\Theta_2\dot{y}(s)\mathrm{d}s\leqslant\\&\frac{1}{\omega_2-\omega_1}\zeta^{\mathrm{T}}(t)\mathcal{Y}_2\Lambda_2\mathcal{Y}_2^{\mathrm{T}}\zeta(t)\end{aligned}\tag{62}$$

Based on Eqs. (16)–(19), for $t > 0$, $\omega(t) = 0$, and $\omega_2 = \omega(t) = \omega_1$, Formulas (61) and (62) still establish Formula (63)

$$\mathcal{L}V_5(y(t),t) \leqslant \frac{1}{2}(\omega_2-\omega_1)^2\dot{y}^{\mathrm{T}}(t)(G_1+G_2)\dot{y}^{\mathrm{T}}(t)- \int_{-\omega_2}^{-\omega_1}\int_{t+\theta}^{t-\omega_1}\dot{y}^{\mathrm{T}}(s)G_1\dot{y}(s)\mathrm{d}s\mathrm{d}\theta- \int_{-\omega_2}^{-\omega_1}\int_{t-\omega_2}^{t+\theta}\dot{y}^{\mathrm{T}}(s)G_2\dot{y}(s)\mathrm{d}s\mathrm{d}\theta = \frac{1}{2}(\omega_2-\omega_1)^2\zeta^{\mathrm{T}}(t)\Gamma_1^{\mathrm{T}}(G_1+G_2)\Gamma_1\zeta(t)- \int_{-\omega_2}^{-\omega(t)}\int_{t+\theta}^{t-\omega(t)}\dot{y}^{\mathrm{T}}(s)G_1\dot{y}(s)\mathrm{d}s\mathrm{d}\theta- (\omega_2-\omega(t))\int_{t-\omega(t)}^{t-\omega_1}\dot{y}^{\mathrm{T}}(s)G_1\dot{y}(s)\mathrm{d}s- \int_{-\omega(t)}^{-\omega_1}\int_{t+\theta}^{t-\omega_1}\dot{y}^{\mathrm{T}}(s)G_1\dot{y}(s)\mathrm{d}s\mathrm{d}\theta- \int_{-\omega_2}^{-\omega(t)}\int_{t-\omega_2}^{t+\theta}\dot{y}^{\mathrm{T}}(s)G_2\dot{y}(s)\mathrm{d}s\mathrm{d}\theta- (\omega(t)-\omega_1)\int_{t-\omega_2}^{t-\omega(t)}\dot{y}^{\mathrm{T}}(s)G_2\dot{y}(s)\mathrm{d}s- \int_{-\omega(t)}^{-\omega_1}\int_{t-\omega(t)}^{t+\theta}\dot{y}^{\mathrm{T}}(s)G_2\dot{y}(s)\mathrm{d}s\mathrm{d}\theta \tag{63}$$

For G_1 and G_2 dependent integral term, there exist matrices U_1, U_2, F_1, and F_2 of appropriate dimensions. From Lemma 2, one gets

$$-\int_{-\omega_2}^{-\omega(t)}\int_{t+\theta}^{t-\omega(t)}\dot{y}^{\mathrm{T}}(s)G_1\dot{y}(s)\mathrm{d}s\mathrm{d}\theta \leqslant \zeta^{\mathrm{T}}(t)\{U_1\psi_1^{\mathrm{T}}(\omega(t))+\psi_1(\omega(t))U_1^{\mathrm{T}}+\frac{1}{2}(\omega_2-\omega(t))^2\times U_1G_1^{-1}U_1^{\mathrm{T}}\}\zeta(t) \tag{64}$$

$$-\int_{-\omega(t)}^{-\omega_1}\int_{t+\theta}^{t-\omega_1}\dot{y}^{\mathrm{T}}(s)G_1\dot{y}(s)\mathrm{d}s\mathrm{d}\theta \leqslant \zeta^{\mathrm{T}}(t)\{U_2\psi_2^{\mathrm{T}}(\omega(t))+\psi_2(\omega(t))U_2^{\mathrm{T}}+\frac{1}{2}(\omega(t)-\omega_1)^2\times U_2G_1^{-1}U_2^{\mathrm{T}}\}\zeta(t) \tag{65}$$

$$-\int_{-\omega_2}^{-\omega(t)}\int_{t-\omega_2}^{t+\theta}\dot{y}^{\mathrm{T}}(s)G_2\dot{y}(s)\mathrm{d}s\mathrm{d}\theta \leqslant \zeta^{\mathrm{T}}(t)\{F_1\psi_3^{\mathrm{T}}(\omega(t))+\psi_3(\omega(t))F_1^{\mathrm{T}}+\frac{1}{2}(\omega_2-\omega(t))^2\times F_1G_2^{-1}F_1^{\mathrm{T}}\}\zeta(t) \tag{66}$$

$$-\int_{-\omega(t)}^{-\omega_1}\int_{t-\omega(t)}^{t+\theta}\dot{y}^{\mathrm{T}}(s)G_2\dot{y}(s)\mathrm{d}s\mathrm{d}\theta \leqslant \zeta^{\mathrm{T}}(t)\{F_2\psi_4^{\mathrm{T}}(\omega(t))+\psi_4(\omega(t))F_1^{\mathrm{T}}+\frac{1}{2}(\omega(t)-\omega_1)^2\times F_2G_2^{-1}F_2^{\mathrm{T}}\}\zeta(t) \tag{67}$$

where $\psi_1^{\mathrm{T}}(\omega(t))=(\omega_2-\omega(t))(e_3-e_6)$, $\psi_2^{\mathrm{T}}(\omega(t))=(\omega(t)-\omega_1)(e_2-e_7)$, $\psi_3^{\mathrm{T}}(\omega(t))=(\omega_2-\omega(t))(e_6-e_4)$, and $\psi_4^{\mathrm{T}}(\omega(t))=(\omega(t)-\omega_1)(e_7-e_3)$, respectively. In the light of Lemma 3, $\begin{bmatrix} G_1 & S \\ * & G_2 \end{bmatrix} \geqslant 0$, for $\omega_1<\omega(t)<\omega_2$, we get

$$-(\omega_2-\omega(t))\int_{t-\omega(t)}^{t-\omega_1}\dot{y}^{\mathrm{T}}(s)G_1\dot{y}(s)\mathrm{d}s- (\omega(t)-\omega_1)\int_{t-\omega(t)}^{t-\omega_1}\dot{y}^{\mathrm{T}}(s)G_2\dot{y}(s)\mathrm{d}s \leqslant \frac{\omega_2-\omega(t)}{\omega(t)-\omega_1}\left(\int_{t-\omega(t)}^{t-\omega_1}\dot{y}(s)\mathrm{d}s\right)^{\mathrm{T}}G_1\left(\int_{t-\omega(t)}^{t-\omega_1}\dot{y}(s)\mathrm{d}s\right)- \frac{\omega(t)-\omega_1}{\omega_2-\omega(t)}\left(\int_{t-\omega_2}^{t-\omega(t)}\dot{y}(s)\mathrm{d}s\right)^{\mathrm{T}}G_2\left(\int_{t-\omega_2}^{t-\omega(t)}\dot{y}(s)\mathrm{d}s\right)= \zeta^{\mathrm{T}}(t)\left[\frac{\omega_2-\omega(t)}{\omega(t)-\omega_1}\psi_5G_1\psi_5^{\mathrm{T}}+\frac{\omega(t)-\omega_1}{\omega_2-\omega(t)}\psi_6G_2\psi_6^{\mathrm{T}}\right]\zeta(t) \leqslant 2\zeta^{\mathrm{T}}(t)\psi_5S\psi_6^{\mathrm{T}}\zeta(t) \tag{68}$$

Noticing Formula (3), similar to Ref. [31], we get

$$\left[k_i(y_i(t))-\phi_i^-y_i(t)\right]\left[k_i(y_i(t))-\phi_i^+y_i(t)\right]\leqslant 0, \quad \left[k_i(y_i(t-\omega(t)))-\phi_i^-y_i(t-\omega(t))\right]\times \left[k_i(y_i(t-\omega(t)))-\phi_i^+y_i(t-\omega(t))\right]\leqslant 0 \tag{69}$$

Furthermore, for any matrices $\aleph=\mathrm{diag}\{q_1,q_2,\ldots,q_n\}$ and $\tilde{\aleph}=\mathrm{diag}\{\tilde{q}_1,\tilde{q}_2,\ldots,\tilde{q}_n\}$, we get

$$0\leqslant -2\sum_{i=1}^{n}q_i\left[k_i(y_i(t))-\phi_i^-y_i(t)\right]\left[k_i(y_i(t))-\phi_i^+y_i(t)\right]=2\zeta^{\mathrm{T}}(t)e_8^{\mathrm{T}}\aleph e_8\zeta(t)+ 2\zeta^{\mathrm{T}}(t)e_1^{\mathrm{T}}L_2\aleph e_8\zeta(t)-2\zeta^{\mathrm{T}}(t)e_1^{\mathrm{T}}L_1\aleph e_1\zeta(t) \tag{70}$$

$$0\leqslant -2\sum_{i=1}^{n}\tilde{q}_i\left[k_i(y_i(t-\omega(t)))-\phi_i^-y_i(t-\omega(t))\right]\times \left[k_i(y_i(t-\omega(t)))-\phi_i^+y_i(t-\omega(t))\right]= -2\zeta^{\mathrm{T}}(t)e_{10}^{\mathrm{T}}\tilde{\aleph}e_{10}\zeta(t)+2\zeta^{\mathrm{T}}(t)e_3^{\mathrm{T}}L_2\tilde{\aleph}e_{10}\zeta(t)- 2\zeta^{\mathrm{T}}(t)e_3^{\mathrm{T}}L_1\tilde{\aleph}e_3\zeta(t) \tag{71}$$

By combining Eqs. (54) and (56)–(71), Formula (72) is true

$$\mathcal{L}V(y(t),m(t))\leqslant \zeta^{\mathrm{T}}(t)\Im(\omega(t),\dot{\omega}(t))\zeta(t)+\int_{t-\omega_1}^{t}y^{\mathrm{T}}(s)(\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)\mathcal{J}_1(\eta)- O_1(\iota))y(s)\mathrm{d}s+\int_{t-\omega(t)}^{t}y^{\mathrm{T}}(s)(\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)\sum_{u=2}^{3}\mathcal{J}_u(\eta)- O_2(\iota))y(s)\mathrm{d}s+\int_{t-\omega_2}^{t-\omega(t)}y^{\mathrm{T}}(s)(\sum_{\eta=1}^{\mathcal{N}}\pi_{\iota\eta}(h)\mathcal{J}_2(\eta)- O_2(\iota))y(s)\mathrm{d}s+\int_{t-\omega_1}^{t}y^{\mathrm{T}}(s)(\sum_{\eta=1}^{\mathcal{N}}\omega_1\pi_{\iota\eta}(h)\times O_1(\eta)y(s)\mathrm{d}s-\Re_1)y(s)\mathrm{d}s+\int_{t-\omega_2}^{t}y^{\mathrm{T}}(s)(\sum_{\eta=1}^{\mathcal{N}}\omega_2\pi_{\iota\eta}(h)\times O_2(\eta)y(s)\mathrm{d}s-\Re_2)y(s)\mathrm{d}s \tag{72}$$

From Formulas (30)–(34) and the Schur's complement lemma, it can be deduced that $\Im(\omega(t),\dot{\omega}(t))<0$, similar to

Ref. [32], define $\Xi = \Im(\omega(t),\dot{\omega}(t))$ and $\nu_1 = \min_{\iota\in\mathcal{N}}\{\lambda_{\min}(-\Xi)\}$ for any $t \geqslant \omega_2$, where $\lambda_{\min}$ is the least eigenvalue

$$\mathcal{L}V(y(t),m(t)) \leqslant -\nu_1\|\zeta(t)\|^2 \leqslant -\nu_1\|y(t)\|^2 \tag{73}$$

and owing to the Dynkin's formula

$$\varepsilon\{V(y(t),m(t)) \mid (\delta,m_0)\} \leqslant \varepsilon\{V(\delta,m_0)\} - \nu_1\varepsilon\left\{\int_0^t \|y(s)\|^2 \mathrm{d}s\right\} \tag{74}$$

thus, based on Eq. (43) for any $t \geqslant 0$, Formula (75) is obtained

$$\varepsilon\{V(y(t),m(t))\} \geqslant \nu_2\varepsilon\left\{\|y(t)\|^2\right\} \tag{75}$$

where $\nu_2 = \min_{\iota\in\mathcal{N}}\{\lambda_{\min}(-P(\iota))\} > 0$. Also from Formulas (74) and (75), Formula (76) can be acquired

$$\varepsilon\left\{\|y(t)\|^2\right\} \leqslant b_2V(y(t),m(t)) - b_1\varepsilon\left\{\int_0^t \|y(s)\|^2\mathrm{d}s\right\},$$
$$b_1 = \nu_1\nu_2^{-1},\ b_2 = \nu_2^{-1} \tag{76}$$

Applying the Gronwell-Bellman lemma gives

$$\varepsilon\left\{\|y(t)\|^2\right\} \leqslant b_2V(x(t),m(t))\mathrm{e}^{-b_1 t} \tag{77}$$

next

$$\varepsilon\left\{\int_0^t \|y(s)\|^2\mathrm{d}s \mid (\delta,m_0)\right\} \leqslant -b_1^{-1}b_2V(y(t),m(t))\times (e^{-b_1 t}-1) \tag{78}$$

Letting $t \to \infty$, we realize

$$\lim_{t\to\infty}\varepsilon\left\{\int_0^t \|y(s)\|^2\mathrm{d}s \mid (\delta,m_0)\right\} \leqslant b_1^{-1}b_2V(y(t),m(t)) \tag{79}$$

then, there is always a scalar $c > 0$, we get

$$\lim_{t\to\infty}\varepsilon\left\{\int_0^t \|y(s)\|^2\mathrm{d}s \mid (\delta,m_0)\right\} \leqslant c \sup_{s\in[-\omega_2,0]} \|\delta(s)\|^2 \tag{80}$$

thus, in view of Definition 2, Eq. (5) is stochastic stable. Theorem 1 is proved. ■

When $u(t) \neq 0$, a state-feedback control law is designed

$$u(t) = -K(m(t))y(t) \tag{81}$$

where $K(m(t))$ is the matrix to be determined, and Eq. (5) can be revised as Eq. (82)

$$\dot{y}(t) = -(D(m(t)) + K(m(t)))y(t) + A(m(t))k(y(t)) + B(m(t))k(y(t-\omega(t))) \tag{82}$$

Similar to Theorem 1, we can get that the closed-loop sMJS in Eq. (82) is stochastically stable, and the proof process is simple and the same as that of Theorem 1.

IV. Illustrative Example

Consider the closed-loop sMJS in Eq. (82) with following parameters

$$D1 = \begin{bmatrix} -1.6 & -1.0 \\ -1.0 & -1.3 \end{bmatrix},\quad D2 = \begin{bmatrix} -0.4036 & -1.0000 \\ -1.0000 & -1.5036 \end{bmatrix},$$
$$A1 = \begin{bmatrix} 1.5 & 0.0 \\ 0.0 & -1.0 \end{bmatrix},\quad A2 = \begin{bmatrix} 1.50 & 0.23 \\ 0.45 & -1.12 \end{bmatrix},$$
$$B1 = \begin{bmatrix} -1.0 & 0.5 \\ -0.5 & -1.0 \end{bmatrix},\quad B2 = \begin{bmatrix} -1.10 & 0.25 \\ -0.50 & -0.10 \end{bmatrix},$$

$$\omega(t) = 1.36 + 0.7\sin(t),\ k_1(\mu) = k_2(\mu) = \tanh(\mu).$$

it is assumed that TR matrix is $[-0.1280, 2.6064; 1.3026, -0.0640]$. Moreover, we set initial condition as $y(t) = [-4,4]^{\mathrm{T}}$. Figure 1 shows the system modes evolution of the semi-Markovian jump process. It can be clearly seen from Fig. 1, the system modes jump between mode 1 and mode 2 randomly. Figures 2 and 3 are the orbits of Eq. (82) even though the system modes are jumping. Especially, Fig. 3 utilizes different initial values to further validate the effectiveness of the designed method. In Fig. 4, the control law curves of Eq. (82) are drawn. Hence, from the aforementioned analyses, Eq. (82) with semi-Markov jump parameters is stochastically stable in mean-square sense.

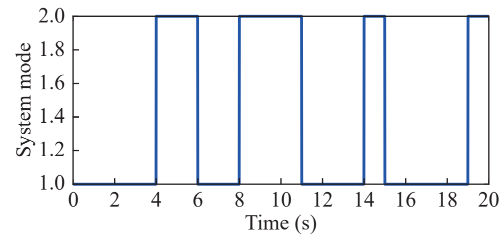

Figure 1 Mode evolution of sMJS.

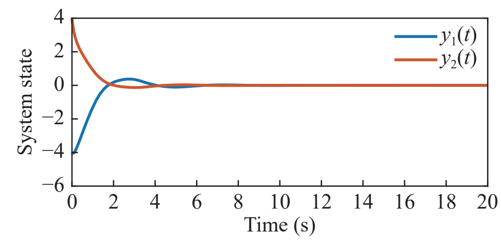

Figure 2 Trajectory of sMJS.

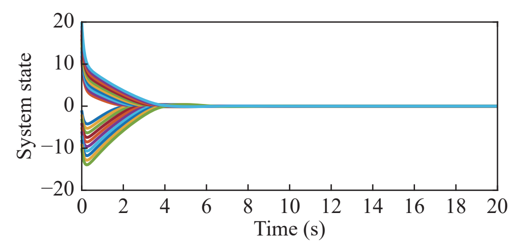

Figure 3 Trajectory of sMJS with different initial values.

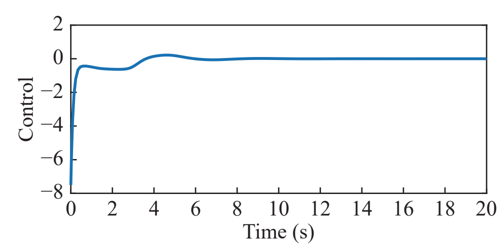

Figure 4 Control input.

V. Conclusion

This paper analyzed the stability of sMJS neural networks. By using the Lyapunov stability theory, the Wirtinger integral inequality, inverted convex combination technology, and generalized double integral inequality, a unique criterion based on linear matrix inequalities is designed to produce neural network to achieve stochastic stability in mean square. In the future, it merits further probe on the issue of other control schemes, such as event-triggered control, event-based control, etc.

ACKNOWLEDGMENT

This work was supported in part by the National Key R&D Program of China (No. 2018YFA0702200), the National Natural Science Foundation of China (No. 61627809), the Liaoning Revitalization Talents Program (No. XLYC1801 005), and the Natural Science Foundation of Liaoning Province of China (No. 2022JH25/10100008).

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