Design of Observer for Discrete-Time Nonlinear Markov Jump Systems with Unknown Inputs

Zhongda Tian

Abstract—**This study explores the design of an unknown input observer for discrete-time nonlinear Markov jump systems based on Lyapunov theory and Takagi-Sugeno fuzzy models. Firstly, the nonlinear system is transformed into a linear system using linear system theory and the Takagi-Sugeno fuzzy model is applied to solve the nonlinear problem. Then, under the conditions where unknown inputs exist in the state equations of the nonlinear Markov jump system, an unknown input observer is proposed so that the error system for state estimation has no relation to the unknown inputs. The feasibility conditions of the unknown input observer steady-state estimation error system are derived to ensure that the estimation error can approach zero. Additionally, linear matrix inequalities are used to address the feasibility problem of unknown parameters. The introduction of relaxation matrices reduces the conservativeness of the conditions. Finally, the proposed unknown input observer is validated using the vehicle lateral dynamics system. The results indicate that the unknown input observer can accurately track the actual state of the system, and the corresponding estimation errors converge to zero.**

***Index Terms*—Discrete-time, nonlinear system, Markov jump system, linear matrix inequality, unknown input observer**

I. INTRODUCTION

THERE are many systems in life and production. In the theoretical research of these systems, the most important step is accurate modeling, which will directly affect the effectiveness of the research results in practical applications. In engineering applications, the operating point of nonlinear systems can change suddenly due to component failures, maintenance, or random changes in the structure of dynamic systems. For the dynamic systems described above, the Markov jump system (MJS) has been proven to be an effective solution strategy [1]. Considering the powerful modeling ability and rich application potential of MJS, many researchers have conducted extensive studies on the MJS, and some rich research results have been obtained [2, 3]. In practice, there are many similar systems. In the case of the solar power plant, there are two weather patterns: cloudy and sunny. Different weather conditions can cause significant changes in the output power of the plant, which is a dual-mode MJS.

In a certain class of MJS, the local transition probabilities are unknown. To address this issue, Ref. [4] proposes a corresponding observer and a sliding mode control strategy based on this observer. At present, there have been many excellent research results on continuous time MJS, which are not only difficult to apply to discrete-time MJS, but also difficult to obtain better control performance. Therefore, the study of discrete-time MJS has attracted the attention of a large number of scholars. Based on the Lyapunov's theory, Wang et al. studied and verified the design of a positive observer for a linear positive MJS when the transition rate part is known [5]. We all know that a linear system is merely an ideal model. To obtain more general results, Zhang et al. designed lower and upper bound observers for real-time system states of continuous nonlinear regularized MJS using Lyapunov correlation theory design time [6]. However, for discrete-time MJS, the corresponding observer design problem has not been completely solved.

The design of observer and filter is a basic and effective method in the field of system control research involving system state reconstruction, state-based fault detection or diagnosis, and has been deeply studied by many scholars [7, 8]. The effectiveness of state estimation for a real system is influenced by model errors and unknown disturbances. The unknown input observer (UIO) can effectively eliminate the unknown interference, which has also been studied and achieved great results [9, 10]. Based on a modified linear parameter varying (LPV) technique, de Oliveira and Pereira designed a new form of UIO for LPV systems to estimate the state and output matrix changes of the LPV system [11]. Assuming that the linear part of the extended system is perceptible over a finite time interval, the authors have conducted relevant discussions on the unknown input and state estimation for a class of nonlinear systems [12]. However, this approach is not suitable for all nonlinear systems. This shows that the scope of application of the method is limited, that is, the Lipschitz constraint is a constraint on the nonlinearity of the system. For more general nonlinear systems, the design of UIO is still a great challenge.

It is an effective method to describe nonlinear systems by transforming them into locally linear systems through smooth

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fusion of membership functions. From the above introduction, it can be seen that locally linear systems can use many theories of linear systems to solve some problems of nonlinear systems. Therefore, the development of a UIO for nonlinear systems, utilizing the Takagi-Sugeno fuzzy (TSF) model framework, represents a significant research direction. The development of observers within the TSF framework may hence be interpreted as addressing the observer synthesis problem for a broader category of nonlinear systems. Xie et al. developed a TSF-based framework for nonlinear systems and derived an extended formulation of this framework, subsequently addressing the observer design for the expanded formulation [13]. When the current premise variables are partially unknown, a control design method for TSF systems based on observation is developed, focusing on the stabilization problem of TSF systems [14]. Although the UIO of the TSF system was designed in this study, the drawback is that its results are only applicable when the output equation of the TSF control system is linear [15]. In a recent study, the authors designed a UIO for TSF systems and discussed fuzzy output equations to reduce observer conservatism by introducing relaxed variables [16]. Inspired by the above studies, Li et al. further determined that the TSF system is an effective means to describe nonlinear systems, and nonlinear objects need to be considered, which can further expand the application scope of the conclusions and improve the corresponding results to some extent [17].

For nonlinear systems, both classical and modern control theories have been intensively studied. First of all, many practical systems are nonlinear. Secondly, the nonlinear character brings many challenges to the application of control theory. In order to more conveniently apply these classical linear system theories to the nonlinear system, within the TSF modeling framework, the nonlinear system admits a representation through a weighted synthesis of locally linear subsystem models. By using TSF model, the nonlinear system is divided into several linear subsystems, which are defined by if-then rules, and finally connected by multiple member functions of the connector. Using other methods for analysis, the TSF model has the characteristics of representing nonlinear systems more effectively, that is, linear systems can describe nonlinear systems. So far, the classical linear system theory can be effectively applied to the local linearization subsystem, and then the control problems of nonlinear systems can be solved from a macro perspective. With the rapid development of nonlinear system research, TSF model has become an effective and reliable modeling method for nonlinear systems [18].

Generally speaking, TSF models can be connected by a set of linear models with appropriate convex weight functions. Inspired by the TSF model, the theory applicable to linear systems is also applicable to nonlinear systems. In their study, the H_∞ filtering issue was investigated for a class of continuous time TSF nonlinear quadratic systems subject to time-varying delays [19].

In Ref. [20], TSF model is applied to nonlinear discrete systems. This model is primarily employed to examine fault estimation and fault-tolerant control challenges in a category of nonlinear discrete-time systems exhibiting time-varying delays and actuator failures. Given that the governing equations of fractional-order systems typically incorporate numerous parameters and nonlinear components, Wang et al. developed an adaptive fuzzy control strategy grounded in the TSF framework to address the stabilization and control challenges of such systems [21].

This study introduces a novel UIO designed for discrete-time nonlinear systems with Markovian jumping parameters. The principal contributions of this work can be summarized as follows:

(1) We consider using TSF model to transform a class of discrete-time nonlinear regular MJSs into a class of TSF systems. The transformed system can be regarded as a form of local linearity and global nonlinearity, thus extending the applicability of the results in the related literature [17].

(2) For the local linear system, a new UIO structure is designed, which is different from the traditional structure. The stability of the estimation error system is analyzed, and the slack variable is introduced to reduce the conservatism of the conclusion.

(3) The designed UIO is evaluated and validated through numerical simulations to confirm its practical applicability and effectiveness.

The remainder of this study is arranged as follows. Section II presents the nonlinear Markov model incorporating fuzzy output equations based on the TSF framework. In Section III, a UIO is constructed for the aforementioned nonlinear MJS, with stability conditions for the error dynamics rigorously established. Numerical case studies in Section IV validate the performance and efficacy of the proposed observer. Section V summarizes the research results and future work.

Notation $\mathbb{R}^n$ is an n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ is a set of $n\times m$ real matrices. O is the null matrix, and I is an identity matrix with compatible dimensions. $\|\cdot\|$ indicates L_2 norm. For a matrix A, A^{T} represents the transpose matrix of A. For the symmetry matrix P, $P>0$ ($\geqslant 0$) and $P<0$ ($\leqslant 0$) indicate that P is positive (semi) definite and negative (semi) definite, respectively. * indicates symmetry.

II. Preliminaries

In a state space, we can choose to use Lyapunov function to judge the stability of linear time-invariant system $\dot{x}(t)=Ax(t)$. If the selected Lyapunov function is $L(x,t)=x^{\mathrm{T}}(t)Px(t)$, the stability condition of the linear time invariant system is that the positive definite symmetric matrix P satisfies $A^{\mathrm{T}}P+PA<0$. Lyapunov functions can also be used for linear and nonlinear complex systems. After the transformation, it can be classified as a linear matrix inequality (LMI) problem by using linear theory. The LMI problem can be transformed into a convex optimization problem. Because the solution is convenient and no parameters need to be adjusted, the analysis and design problem of the control system can be transformed into the LMI solution problem, which brings convenience to the application of the method [22].

The Lyapunov function can also be used in linear and nonlinear complex systems. After the transformation, processing it using linear theory methods, it can finally be classified as a linear matrix inequality problem. LMI can be transformed into convex optimization problems. Due to the convenience of solving, without adjusting parameters, the analysis and design problems of the control system can be transformed into LMI solving problems, which brings convenience to the application of this method. The inequality shown in Formula (1) is LMI or strict LMI.

$$G(x) = G_0 + \sum_{k=1}^{L} x_k G_k < 0 \tag{1}$$

The unknown variables in the inequality are $x = (x_1, x_2, ..., x_n)^{\mathrm{T}} \in \mathbb{R}^n$ and $G_k = G_k^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $k = 0, 1, \dots, L$ is a given matrix. $G(x) > 0$ denotes that $G(x)$ is positive definite. That is, $u^{\mathrm{T}} R u > 0$ is true for any nonzero vector $u \in \mathbb{R}^n$. When Formula (2) holds, it is non-strict LMI.

$$G(x) = G_0 + \sum_{k=1}^{K} x_k G_k \leqslant 0 \tag{2}$$

Clearly, one LMI can represent multiple LMIs. That is, $G_1(x) > 0$, and $G_2(x) > 0$, $G_3(x) > 0, \dots, G_k(x) > 0$ are equivalent to the following matrix.

$$\begin{bmatrix} G_1(x) & 0 & \cdots & 0 \\ 0 & G_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_k(x) \end{bmatrix} > 0 \tag{3}$$

The LMI in Formula (1) can be regarded as a convex constraint on the variable x, and the set is a convex set, the problem of solving the LMI in Formula (2) can be efficiently formulated as a convex optimization problem. By the Schur complement lemma, matrix inequalities can be transformed from quadratic form to linear form. The biggest advantage of LMI is that it is simple to calculate and can save computing time.

The TSF modeling framework utilizes fuzzy membership functions to achieve smooth interpolation between local linear subsystems, thereby enabling the representation of complex nonlinear systems through a set of locally defined linear models. Fuzzy linear systems can be described using if-then rules. In the total probability space, TSF models with r rules can represent the global model of a class of discrete-time nonlinear MJSs. For TSF models, follow rule i.

Mode rule *i*: If $\varepsilon_1(k) = M_{i1}, M_{i2}, \dots, \varepsilon_l(k) = M_{il}$, then

$$\begin{cases} x(t+1) = A_i(\theta_z)x(t) + B_i(\theta_z)u(t) + D(\theta_z)d(t), \\ y(t) = C_i(\theta_z)x(k),\ i \in \{1, 2, \dots, r\}; \\ x(0) = x_0 \end{cases} \tag{4}$$

where $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, and $u(k) \in \mathbb{R}^{n_u}$ are system state, control input, and system output vector, respectively. $d(t) \in \mathbb{R}^{n_d}$ represents the perturbation and uncertainty in the model, and $F_u = \{1, 2, \dots, r\}$ represents the if-then rule set. $\varepsilon_1(t), \varepsilon_2(t), \dots, \varepsilon_l(t)$ denote the premise variable with quantity l, and M_{ij} ($i \in F_u$, $j = 1, 2, \dots, l$) can be viewed as a fuzzy set. The value of a jump process θ_t is a finite set $S = \{1, 2, \dots, N\}$, which can be represented as a Markov chain.

The mode transition probability is $p_{ij} = P\{\theta_{z+1} = j | \theta_z = i\}$, and for $\forall i \in S$, $\sum_{j=1}^{N} \theta_{ij} = 1$, $A_i(\theta_z) \in \mathbb{R}^{n_x\times n_x}$, $B_i(\theta_z) \in \mathbb{R}^{n_x\times n_u}$. Both $C_i(\theta_z) \in \mathbb{R}^{n_y\times n_x}$ and $D(\theta_z) \in \mathbb{R}^{n_x\times n_d}$ are matrices with known real constants, and both have appropriate dimensions. For ease of presentation, when $\theta_z = z \in S$, $A_i(\theta_z)$, $B_i(\theta_z)$, $C_i(\theta_z)$, and $D(\theta_z)$ can be expressed as A_{iz}, B_{iz}, C_{iz}, and D_z, respectively.

Equation (5) gives the global model of TSF MJS with the help of the standard fuzzy model.

$$\begin{cases} x(t+1) = \sum_{i=1}^{r} h_i(\varepsilon(t)) \left[A_{iz}x(t) + B_{iz}u(t) + D_z d(t)\right], \\ y(t) = \sum_{i=1}^{r} h_i(\varepsilon(t)) C_{iz} x(t) \end{cases} \tag{5}$$

where $k \in S$, $h_i(\varepsilon(t)) = \omega_i(\varepsilon(t)) / \sum_{i=1}^{r} \omega_i(\varepsilon(t))$, $\omega_i(\varepsilon(t)) = \prod_{j=1}^{l} M_{ij}(\varepsilon_j(t))$, and $\varepsilon(t) = \left[\varepsilon_1(t)\varepsilon_2(t)\cdots\varepsilon_l(t)\right]^{\mathrm{T}}$. M_{ij} is a fuzzy set, and $\varepsilon_j(t)$ ($j = 1, 2, \dots, l$) is its membership degree. $\omega_i(\varepsilon(k))$ is the weight of the i-th rule. And, $\sum_{i=1}^{r} h_i(\varepsilon(t)) = 1$, $0 \leqslant h_i(\varepsilon(t)) \leqslant 1$.

Similarly, for a more concise representation, the global model in Eq. (5) can be expressed as

$$\begin{cases} x(t+1) = A_{iz}(h)x(t) + B_{iz}(h)u(t) + D_z d(t), \\ y(t) = C_{iz}(h)x(t) \end{cases} \tag{6}$$

where

$$\begin{cases} A_{iz}(h) = \sum_{i=1}^{r} h_i(\varepsilon(z)) A_{iz}, \\ B_{iz}(h) = \sum_{i=1}^{r} h_i(\varepsilon(z)) B_{iz}, \\ C_{iz}(h) = \sum_{i=1}^{r} h_i(\varepsilon(z)) C_{iz} \end{cases} \tag{7}$$

In this study, matrix D_z is assumed as a column-full-rank matrix and satisfies Eq. (8).

$$\operatorname{rank}\begin{pmatrix} I & D \\ C_{iz}(h^+) & 0 \end{pmatrix} = n_x + n_d \tag{8}$$

where $h^+ = (h_1(\varepsilon(t+1)), h_2(\varepsilon(t+1)), ..., h_r(\varepsilon(t+1)))$.

This paper aims to design the UIO shown in Eq. (6) such that it can satisfy the state estimation error of a discrete-time nonlinear MJS independent of the unknown input $d(k)$.

Lemma 1 Schur complement lemma S_1, S_2, and S_3 are constant matrices, $S_1 = S_1^{\mathrm{T}}$ and $0 < S_2 = S_2^{\mathrm{T}}$, then $S_1 + S_3^{\mathrm{T}} S_2^{-1} S_3 < 0$. Given constant matrices S_1, S_2, and S_3, where $S_1 = S_1^{\mathrm{T}}$ and $0 < S_2 = S_2^{\mathrm{T}}$, then $S_1 + S_3^{\mathrm{T}} S_2^{-1} S_3 < 0$. If and only if $\begin{bmatrix} S_1 & S_3^{\mathrm{T}} \\ S_3 & -S_2 \end{bmatrix} < 0$ or $\begin{bmatrix} -S_2 & S_3 \\ S_3^{\mathrm{T}} & S_1 \end{bmatrix} < 0$.

III. Main Results

For the nonlinear MJS in Eq. (6), the designed UIO is

$$\begin{cases} \xi(t+1) = T(h^+)A_{iz}(h) + \\ \quad T(h^+)B_{iz}(h)u(t) + L(h)(y(t) - C_{iz}(h)\hat{x}(t)), \\ \hat{x}(t) = \xi(t) + Ny(t) \end{cases} \tag{9}$$

where $\xi(t) \in \mathbb{R}^{n_x}$ denotes the intermediate variable and $\hat{x}(t)$ is the estimated state variable of $N \in \mathbb{R}^{n_x \times n_y}$. $L(h) \in \mathbb{R}^{n_x \times n_y}$ and $T(h^+) \in \mathbb{R}^{n_x \times n_x}$ are the matrices to be designed. $L(h)$ and $T(h^+)$ can be found in Eq. (10).

$$\begin{cases} L(h) = \sum_{i=1}^{r} h_i(\varepsilon(t))L_i, \\ T(h^+) = \sum_{i=1}^{r} h_i(\varepsilon(t+1))T_i \end{cases} \tag{10}$$

In addition, the matrices $T(h^+)$ and N should conform to the following two equations.

$$T(h^+) + NC_{iz}(h^+) = I \tag{11}$$

$$T(h^+)D_z = 0 \tag{12}$$

The following theorem is proposed to synthesize the UIO of Eq. (9).

Theorem 1 The state of the estimated UIO in Eq. (9) converges asymptotically to the state of Eq. (6) if there are matrices $T(h^+)$ and N satisfying the settings in Eqs. (11) and (12), and for any $h = [h_1(\varepsilon(t)), h_2(\varepsilon(t)), \ldots, h_r(\varepsilon(t))]$ and $h^+ = [h_1(\varepsilon(t+1)), h_2(\varepsilon(t+1)), \ldots, h_r(\varepsilon(t+1))]$, there are matrices $W(h) \in \mathbb{R}^{n_x \times n_y}$ and $P = P^{\mathrm{T}} \in \mathbb{R}^{n_x \times n_x} > 0$.

$$\Phi = \begin{bmatrix} -P & * \\ PT(h^+)A_{iz}(h) - W(h)C_{iz}(h) & -P \end{bmatrix} < 0 \tag{13}$$

In addition, if Formula (13) is true, then $L(h) = P^{-1}W(h)$ is the solution of $L(h)$ in the UIO designed by Eq. (9).

Proof Define state observation error $e(t)$, which can be expressed as $e(t) = x(t) - \hat{x}(t)$. The following can be obtained.

$$\begin{aligned} e(t+1) &= x(t+1) - \hat{x}(t+1) = x(t+1) - \xi(t+1) - \\ & Ny(t+1) = x(t+1) - [T(h^+)A_{iz}(h)\hat{x}(t) + \\ & T(h^+)B_{iz}(h)u(t) + L(h)(C_{iz}(h)x(t) - \\ & C_{iz}(h)\hat{x}(t))] - NC_{iz}(h^+)x(k+1) = \\ & [I - NC_{iz}(h^+)]x(t+1) - T(h^+)A_{iz}(h)\hat{x}(t) - \\ & T(h^+)B_{iz}(h)u(t) - L(h)C_{iz}(h)e(t) = \\ & T(h^+)[A_{iz}(h)x(t) + B_{iz}(h)u(t) + D_z d(t)] - \\ & T(h^+)A_{iz}(h)\hat{x}(t) - T(h^+)B_{iz}(h)u(t) - \\ & L(h)C_{iz}(h)e(t) = T(h^+)[A_{iz}(h)(e(t) + \hat{x}(t)) + \\ & B_{iz}(h)u(t) + D_z d(t)] - T(h^+)A_{iz}(h)\hat{x}(t) - \\ & T(h^+)B_{iz}(h)u(t) - L(h)C_{iz}(h)e(t) = \\ & [T(h^+)A_{iz}(h) - L(h)C_{iz}(h)]e(t) + \\ & [T(h^+)D_z]d(t). \end{aligned}$$

Then, if Eq. (9) holds, Eq. (14) can be obtained.

$$e(t+1) = [T(h^+)A_{iz}(h) - L(h)C_{iz}(h)]e(t) \tag{14}$$

It follows from the above that if the matrices $T(h^+)$ and $L(h)$ can stabilize the error system in Eq. (14), the estimation error given by Eq. (14) can converge to zero, which implies a stable error system. Based on Lyapunov stability theory, the error system is analyzed in detail.

The candidate Lyapunov function $V(t) = e^{\mathrm{T}}(t)Pe(t)$ is chosen. For $\Delta V(t)$, the change between each adjacent sampling interval is $V(t)$, then we have the following equation.

$$\begin{aligned} &\Delta V(t) = V(t+1) - V(t) = \\ &e^{\mathrm{T}}(t+1)Pe(t+1) - e^{\mathrm{T}}(t)Pe(t) = \\ &e^{\mathrm{T}}(t)\{[T(h^+)A_{iz}(h) - L(h)C_{iz}(h)]^{\mathrm{T}} P[T(h^+)A_{iz}(h) - \\ &L(h)C_{iz}(h)] - P\}e(t), \end{aligned}$$

$\Delta V(t) < 0$ means that

$$\begin{aligned} &[T(h^+)A_{iz}(h) - L(h)C_{iz}(h)]^{\mathrm{T}} P[T(h^+)A_{iz}(h) - \\ &L(h)C_{iz}(h)] - P < 0 \end{aligned} \tag{15}$$

The Schur complement lemma in Lemma 1 shows that Formula (15) is consistent with the following LMI.

$$\begin{bmatrix} -P & * \\ P[T(h^+)A_{iz}(h) - L(h)C_{iz}(h)] & -P \end{bmatrix} < 0 \tag{16}$$

Let $W(h) = PL(h)$, we can obtain Formula (13). The proof of Theorem 1 is complete. ■

Remark 1 It should be noted that the conditional integrals of Eqs. (8) and (9) are

$$\begin{bmatrix} T(h^+) & N \end{bmatrix} \begin{bmatrix} I & D_z \\ C_{iz}(h^+) & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \tag{17}$$

If Eq. (17) is feasible, then the feasible solutions T_i, $i = 1, 2, ..., r$ and N can be solved by the following algorithm.

$$\begin{cases} \begin{bmatrix} T_i & N \end{bmatrix} \begin{bmatrix} I & D_z \\ C_{iz} & 0 \end{bmatrix} - \begin{bmatrix} I & 0 \end{bmatrix} \geqslant 0, \\ \begin{bmatrix} T_i & N \end{bmatrix} \begin{bmatrix} I & D_z \\ C_{iz} & 0 \end{bmatrix} - \begin{bmatrix} I & 0 \end{bmatrix} \leqslant 0, \end{cases} \tag{18}$$

$$i = 1, 2, \ldots, r$$

Importantly, UIO cannot be designed directly from the condition in Formula (13). Accordingly, the subsequent theorem establishes the equivalence between the stability criteria presented in Theorem 1 and a set of LMI constraints.

Theorem 2 If the matrices T_i, $i \in F_u$, and N exist, consider Eqs. (6) and (9), and the conditions in Formula (18) are satisfied, then for all $i, j, l \in F_u$, there are matrices $W(h) \in \mathbb{R}^{n_x \times n_y}$ and $0 < P = P^{\mathrm{T}} \in \mathbb{R}^{n_x \times n_x}$ satisfying Formula (19).

$$\Phi_{ijl} = \begin{bmatrix} -P & * \\ PT_l A_{iz} - W_j C_{iz} & -P \end{bmatrix} < 0 \tag{19}$$

Then the estimation error system in Eq. (14) is acceptable. If condition in Formula (19) is feasible, then Eq. (20) gives a solution to the parameter $L(h)$ of UIO in Eq. (9).

$$L(h) = P^{-1}W(h) = P^{-1}\left(\sum_{i=1}^{r} (\varepsilon(t))W_i\right) \tag{20}$$

Proof Define the following function

$$W(h) = \sum_{i=1}^{r} h_i(\varepsilon(t))W_i \tag{21}$$

Combining Eqs. (7), (10), and (21), the following can be obtained

$$\Phi = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(\varepsilon(t))h_j(\varepsilon(t))h_l(\varepsilon(t+1))\Phi_{ijl}.$$

Thus, we have $\Phi < 0$ if the condition of Formula (19) is satisfied. The proof of Theorem 2 is complete. ■

Using the method in Ref. [23], the conservatism associated with UIO in Eq. (9) is addressed in the theorem presented below.

Theorem 3 Consider a nonlinear MJS in Eq. (6), if the existence of matrices T_i, $i \in F_u$, and N satisfies the conditions in Formula (18) and for all $i, j, l \in F_u$, there exist matrices $W_i \in \mathbb{R}^{n_x \times n_y}$, $U_{ijl} \in \mathbb{R}^{3n_x \times 3n_x}$, and $U_{iil} = U_{iil}^{\mathrm{T}} \in \mathbb{R}^{3n_x \times 3n_x}$, and $0 < P = P^{\mathrm{T}} \in \mathbb{R}^{n_x \times n_x}$ exists, then the following variety is obtained.

$$\Phi_{iil} < U_{iil},\ i,\ l \in F_u \tag{22a}$$

$$\Phi_{ijl} + \Phi_{jil} < U_{ijl} + U_{ijl}^{\mathrm{T}},\ 1 \leqslant i < j \leqslant r,\ l \in F_u \tag{22b}$$

$$\begin{bmatrix} U_{11l} & U_{12l} & \cdots & U_{1rl} \\ U_{12l}^{\mathrm{T}} & U_{22l} & \cdots & U_{2rl} \\ \vdots & \vdots & \ddots & \vdots \\ U_{1rl}^{\mathrm{T}} & U_{2rl}^{\mathrm{T}} & \cdots & U_{rrl} \end{bmatrix} < 0,\ l \in F_u \tag{22c}$$

where Φ_{ijl} is defined in Theorem 2, then there exists UIO in Eq. (9) that decouples the estimated dynamic error system in Eq. (14) from the disturbance and gradually converges to zero. Moreover, when the conditions in Formulas (22a)–(22c) are satisfied, the parameters of the designed UIO can be obtained from Eq. (20).

Proof According to Theorem 2, we get

$$\Phi = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(\varepsilon(t))h_j(\varepsilon(t))h_l(\varepsilon(t+1))\Phi_{ijl} =$$
$$\sum_{i=1}^{r}\sum_{l=1}^{r} h_i^2(\varepsilon(t))h_l(\varepsilon(t+1))\Phi_{iil}+$$
$$\sum_{i=1}^{r}\sum_{j=i+1}^{r}\sum_{l=1}^{r} h_i(\varepsilon(t))h_j(\varepsilon(t))h_l(\varepsilon(t+1))\times(\Phi_{ijl}+\Phi_{jil}).$$

If conditions in Formulas (22a)–(22c) are true, then

$$\Phi < \sum_{i=1}^{r}\sum_{l=1}^{r} h_i^2(\varepsilon(t))h_l(\varepsilon(t+1))U_{iil}+$$
$$\sum_{i=1}^{r}\sum_{j=i+1}^{r}\sum_{l=1}^{r} h_i(\varepsilon(t))h_j(\varepsilon(t))h_l(\varepsilon(t+1))\times(U_{ijl}+U_{ijl}^{\mathrm{T}}),$$

$$\Phi < \sum_{l=1}^{r} h_l(\varepsilon(t+1))\times \begin{bmatrix} h_1 I \\ h_2 I \\ \vdots \\ h_r I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} U_{11l} & U_{12l} & \cdots & U_{1rl} \\ U_{12l}^{\mathrm{T}} & U_{22l} & \cdots & U_{2rl} \\ \vdots & \vdots & \ddots & \vdots \\ U_{1rl}^{\mathrm{T}} & U_{2rl}^{\mathrm{T}} & \cdots & U_{rrl} \end{bmatrix} \begin{bmatrix} h_1 I \\ h_2 I \\ \vdots \\ h_r I \end{bmatrix} \tag{23}$$

If Formula (22c) holds, then $\Phi < 0$ holds. The proof of Theorem 3 is complete. ■

Remark 2 Compared with Theorem 2, the conservatism of the original theorem is reduced by introducing slack variables in Theorem 3. In Ref. [23], the applicability of this method has been demonstrated.

IV. Verification

This section provides a case study to validate the effectiveness of the designed UIO. Given a vehicle lateral dynamics system, the system description is detailed in Refs. [24, 25].

$$\begin{cases} \begin{bmatrix} \dot{\beta}(k) \\ \dot{r}(k) \end{bmatrix} = \begin{bmatrix} -\dfrac{c_{\alpha V}+c_{\alpha H}}{m v_{ref}} & \dfrac{l_H c_{\alpha H} - l_l c_{\alpha V}}{m v_{ref}^2} - 1 \\ \dfrac{l_H c_{\alpha H} - l_V c_{\alpha V}}{I_z} & -\dfrac{l_V^2 c_{\alpha V} + l_H^2 c_{\alpha H}}{I_z v_{ref}} \end{bmatrix} \begin{bmatrix} \beta(k) \\ r(k) \end{bmatrix} + \\ \begin{bmatrix} \dfrac{c_{\alpha V}}{m v_{ref}} \\ \dfrac{l_V c_{\alpha V}}{I_z} \end{bmatrix} u(k), \\ y(k) = \alpha_y(k) - \dfrac{c_{\alpha V}}{m} u(k) = -\dfrac{c_{\alpha V}+c_{\alpha H}}{m}\beta(k) + \\ \quad \dfrac{l_H c_{\alpha H} - l_V c_{\alpha V}}{m v_{ref}^2} r(k), \end{cases}$$

where $\beta(k)$ represents the sideslip angle of the vehicle, $r(k)$ represents the yaw angle, $u(k)$ is the angle of the steering wheel, $\alpha_y(k)$ represents the lateral acceleration, v_{ref} is a group of vehicle speeds, $m = 991$ kg is the total mass of the vehicle, $c_{\alpha V} = 41{,}600$ N/rad is the steering stiffness of the front wheel, $c_{\alpha H} = 47{,}130$ N/rad is the steering stiffness of the rear wheel, $l_V = 1$ m represents the distance from the center of gravity of the vehicle front axle, $l_H = 1.46$ m represents the distance from the center of mass of the vehicle to the rear axle, and $I_z = 1574\ \mathrm{kg \cdot m^2}$ represents the moment of inertia of the z axis.

Based on assumption $\varepsilon(k) = 1/v_{ref} \in [0.0182, 0.2]$, Euler discretization algorithm with sampling interval $\Delta t = 0.01$ s is used. The fuzzy model can be approximately equivalent to the above system.

Rule 1: If $\varepsilon_1(t)$ is about 0.0182, then

$$\begin{cases} x(t+1) = A_{1z}(h)x(t) + B_{1z}(h)u(t) + D_z d(t), \\ y(t) = C_{1z}(h)x(t). \end{cases}$$

Rule 2: If $\varepsilon_1(t)$ is about 0.2, then

$$\begin{cases} x(t+1) = A_{2z}(h)x(t) + B_{2z}(h)u(t) + D_z d(t), \\ y(t) = C_{2z}(h)x(t). \end{cases}$$

The followings are the parameters of the discrete-time nonlinear MJS.

$$A_{1z} = \begin{bmatrix} 0.9837 & -0.0099 \\ 0.1729 & 0.9836 \end{bmatrix},\ A_{2z} = \begin{bmatrix} 0.9374 & -0.0003 \\ 0.1729 & 0.8195 \end{bmatrix},$$

$$B_{1z} = \begin{bmatrix} 0.0076 \\ 0.2643 \end{bmatrix},\ B_{2z} = \begin{bmatrix} 0.0785 \\ 0.2643 \end{bmatrix},\ C_{1z} = \begin{bmatrix} -89.5358 & 0.0091 \end{bmatrix},$$

$$C_{2z} = \begin{bmatrix} -83.7075 & 1.0268 \end{bmatrix},\ D_{1z} = D_{2z} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The motion trajectory of $\varepsilon(t)$ in the simulation is

$$\begin{cases} 0.0182,\ t < 10\ ; \\ 0.0182 + \dfrac{0.2 - 0.0182}{80 - 10},\ 10 \leqslant t \leqslant 80\ ; \\ 0.2,\ t > 80\ . \end{cases}$$

The membership functions $h_1(\varepsilon(k))$ and $h_2(\varepsilon(k))$ are given below.

$$\begin{cases} h_1(\varepsilon(t)) = \dfrac{0.2 - \varepsilon(t)}{0.2 - 0.0182}, \\ h_2(\varepsilon(t)) = \dfrac{\varepsilon(t) - 0.0182}{0.2 - 0.0182}. \end{cases}$$

The YALMIP toolbox is used to solve Formula (18), yielding solutions to Eq. (14) as

$$T_{1z} = \begin{bmatrix} 0 & 0.0001 \\ 0 & 1 \end{bmatrix},\quad N_{1z} = \begin{bmatrix} -0.0112 \\ 0 \end{bmatrix},$$
$$T_{2z} = \begin{bmatrix} 0 & 0.0123 \\ 0 & 1 \end{bmatrix},\quad N_{2z} = \begin{bmatrix} -0.0119 \\ 0 \end{bmatrix}.$$

Furthermore, the solution of Formula (19) can be obtained through the LMI toolbox, and the matrix L_{iz} of the following equations can be obtained.

$$L_{1z} = \begin{bmatrix} -0.0001 \\ -0.0019 \end{bmatrix},\quad L_{2z} = \begin{bmatrix} -0.0001 \\ -0.0019 \end{bmatrix}.$$

The mode of the given system is shown in Fig. 1, and the corresponding transition probability matrix is $p_{12} = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$.

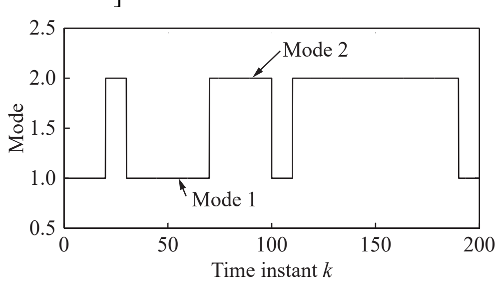

Fig. 1 The simulation of system mode.

Fig. 1 shows the process of the system randomly jumping between different modes, following a set of Markov chains. The unknown input is determined as $d(t) = \begin{cases} 0.5e^{-0.1z}\sin(z),\ 0 < z < 20 \\ 0.5e^{-0.1z}\sin(z) + \varDelta,\ 20 \leqslant z \end{cases}$, where $\varDelta \in [-1, 1]$. The control input is set to a fixed value $u(t) = 0.1$, the initial state of the system is $\hat{x}(0) = [\ 0 \quad 0\ ]^{\mathrm{T}}$, and the estimated initial state is $\hat{x}(0) = [\ 0 \quad 0\ ]^{\mathrm{T}}$.

Fig. 2 shows the actual values and corresponding estimated values of system state $x_1(t)$. The results in the figure are: the black solid line represents the actual state value of the system, and the yellow dashed line represents the estimated state value of the system.

Similarly, Fig. 3 shows the actual values and corresponding estimated values of system state $x_2(t)$. The results in the figure are: the black solid line represents the actual state value of the system, and the yellow dashed line represents the estimated state value of the system.

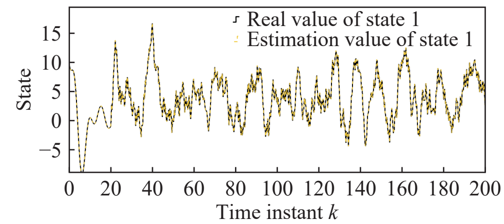

Fig. 2 System state and estimated state ($x_1(t)$).

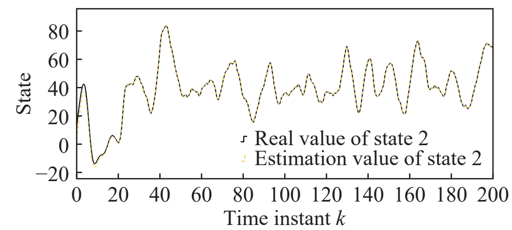

Fig. 3 System state and estimated state ($x_2(t)$).

For the two systems $x_1(t)$ and $x_2(t)$, Fig. 4 illustrates the estimation error between the actual values and the estimated values. $e_1(t) = \hat{x}_1(t) - x_1(t)$ is the estimation error of state $x_1(t)$, and $e_2(t) = \hat{x}_2(t) - x_2(t)$ is the estimation error of state $x_2(t)$. In Fig. 4, the estimation error of system state 1 is represented by a solid red line, and that of system state 2 is represented by a dashed blue line.

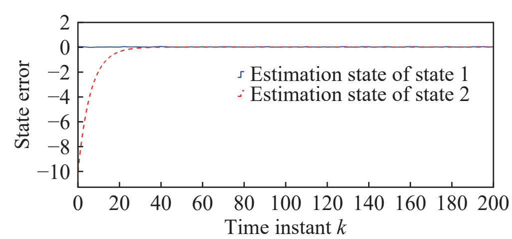

Fig. 4 System state estimation error.

From the above comparison results, it can be seen that the estimation error of the designed UIO can converge to zero, indicating that after multiple samplings, this UIO can effectively estimate the lateral slip angle and yaw velocity of the vehicle.

V. Conclusions

This paper designs a UIO of discrete MJS with a fuzzy output equation. Based on the design method of Lyapunov functions, a new UIO structure was derived. Transform the design conditions of UIO into a set of LMIs and solve them using the LMI toolbox in Matlab. The feasibility conditions of the designed UIO were verified. The practicability and effectiveness of the vehicle lateral dynamics system were verified through simulation. Simulation results demonstrate that the proposed UIO achieves accurate estimation of the system states.

For future research, the proposed UIO holds potential for implementation in controller synthesis and fault diagnosis schemes, with the aim of enhancing both control performance and detection accuracy. Furthermore, as this study proposes a novel UIO structure, the observer design for discrete-time systems can be extended to continuous-time systems, positive systems, etc., to achieve better results or further reduce conservation.

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